Incomplete-Market Equilibria Solved Recursively on an Event Tree

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Incomplete markets show a lot of promise

- Because of
  - the large size of non traded human capital, causing idiosyncratic risk not to be traded,
  - financial markets in the real world are massively incomplete
- Missing-market risks "rock the boat" of traded markets, increasing risk premia and volatility
- Macroeconomists have already steered in that direction
- Part of a broader class of equilibrium problems, which will be handled similarly

Our objective

- Definition of Walrasian equilibrium: a set of prices and decisions such that every agent makes optimal decisions taking prices as given, and supply equals demand in every market
- We develop a method that allows one to compute incomplete-market equilibria routinely (when they exist) on an event tree
- We prefer a recursive approach as being less likely to go astray numerically, over a "global" approach, which would solve all first-order conditions of all times and states simultaneously
- "Dual" approach: we focus on calculating state prices. *Final result is a recursive construction of tomorrow's individual state prices as functions of today's state prices.*
  - or, equivalently, of *tomorrow's individual consumptions as functions of today's distribution of consumption in the population.*

The main difficulty to be overcome

State variables:

- there are exogenous state variables driving the economy (say, output).
  - These are used to build the tree forward.
  - Note: our technique is not based on a Markovian assumption (but, in the nonMarkov case, the tree becomes "bushy".)
- but, in an incomplete market, there are also *endogenous state variables* (say, to fix ideas, "the distribution of wealth" in the population; but we use state prices instead; see later)
- The system is “forward-backward”.

"Traditional/primal" recursive approach: dynamic program with tatonnement over the process of state variables

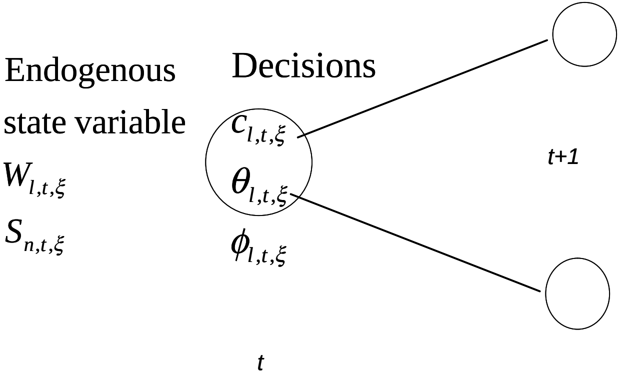
(den Haan (1994, 1997), Krusell and Smith (1997, 1998), Storesletten, Telmer and Yaron (2007)):

1. Let endogenous, agent-specific state variables include accumulated wealth and asset prices (as in Merton (1969)).
2. Postulate agents' beliefs or forecasts in the form of some dynamics for the state variables
3. Solve each investor's consumption-portfolio choice problem by dynamic programming, using either value-function iteration and either first-order conditions or exhaustive enumeration
4. Simulate the resulting economy, equating demand with supply to determine the values of the endogenous variables
5. Run linear regressions to estimate the dynamics of the state variables and thereby model agents' forecasts. Stop if the dynamics so calculated are close to those postulated under 2. Otherwise, go back to 3.

Our technique brings several key benefits

1. The system is entirely backward all the way to time 0.
2. Wealth accumulated from the past is not a state variable:
    - domain not known a priori: how to guarantee that each agent has enough wealth remaining for the equilibrium to exist?
    - when it is used as a state variable, one has to limit artificially the domain of agents' wealth
    - and, for that, one has to limit the positions they can take
    - we never have to limit the positions taken by agents because we do not used wealth or positions as state variables
3. No forecasting step: future consumption decisions are made today. The future is a part of the solution.
4. The algorithm is not limited to a relatively small number of assets. That number only increases the size of the equation system to be solved at each node.
5. We never have to take a derivative of a function that has been interpolated.

"Primal" formulation: first-order conditions



When investor l is faced (in state $\xi \in \mathbb{F}_t$) with entering wealth $W_{l,t,\xi}$, local price vector S_t, and new endowment $e_{l,t,\xi}$, he computes his immediate consumption $c_{l,t,\xi}$ (or his local *Arrow-Debreu shadow prices* $\phi_{l,t,\xi} \in \mathbb{R}$), his immediate trading strategy $\theta_{l,t,\xi}$ subject to flow budget constraint:

$$c_{l,t,\xi} + \theta_{l,t,\xi} \cdot S_t = e_{l,t,\xi} + W_{l,t,\xi}$$

Dual approach

Accumulated wealth is not a "good" state variable because it is not known a priori what its domain is.

Further, value functions need to be interpolated and differentiated: numerically bad.

- Switching to the dual approach, we can use state prices or consumption as endogenous state variables and do away with the value functions.
- However, the resulting system of FOCs cannot be solved recursively in the backward way
  - because the unknowns at time t include consumptions at time t, $c_{l,t}$,
  - whereas the portfolio-choice first-order conditions would involve consumptions at time $t+1$, which at time t would already be solved for.
  - In this form, the system is simultaneously forward and backward at each point in time. We want to re-formulate it so that it can be solved backward all the way back to time 0.

Making a recursive solution possible

- We *first* re-cast the flow-budget condition (the second FOC condition) in terms of agent l's wealth *exiting* period t, which is simply

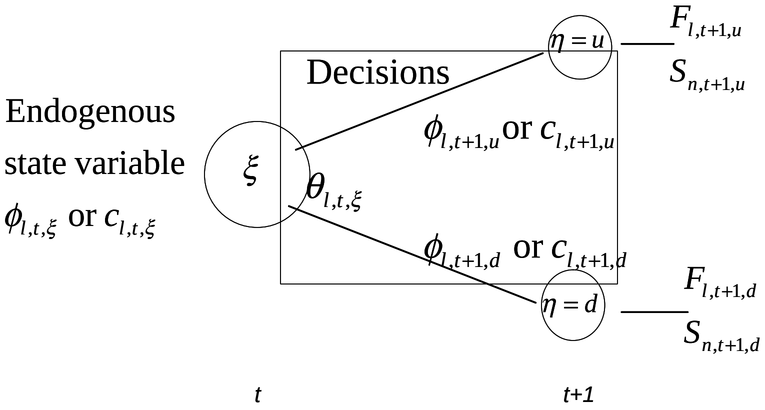
$$F_{l,t} \triangleq \theta_{l,t} \cdot S_t$$

- As the wealth entering period t is $W_{l,t} = \theta_{l,t-1} \cdot (S_t + \delta_t)$, the flow-budget constraint (which will be used at time $t+1$) – a "marketability condition" –, can be written equivalently as

$$c_{l,t,\xi} + F_{l,t,\xi} = e_{l,t,\xi} + \theta_{l,t-1,\xi^-} \cdot (S_t + \delta_t)$$

Making a recursive solution possible

Secondly, we introduce a crucial *time-shift of equations*, or regrouping. We combine consumption FOCs and budget constraints for period $t + 1$ with portfolio FOC for period t.



Example: the Basak-and-Cuoco (1998) equilibrium

- Agents of Category 1 ("shareholders") receive an endowment stream e which follows a geometric Brownian motion. We capture that endowment with a re-combining binomial tree with fixed drift and volatility.
- Agents of Category 2 ("non shareholders") receive no endowment stream but they are able to consume because they start their lives with some initial financial claims W_0 on Agents of Category 1:
- The market is incomplete; the only traded security is an instantaneously riskless one.
- This economy is formally identical to the limited-participation economy of Basak and Cuoco (1998), except for a small difference in interpretation. Furthermore,
  - they study the log-utility case
  - we generalize to any utility function
- This is the simplest of our examples: only the backward induction of wealth function is needed. No functions for asset prices need be carried backward.

Nodal system of equations for the Basak-Cuoco example

Flow budget constraints or "marketability" conditions

$$c_{1,t+1,u} + F_{1,t+1,u}\left(\right) = \theta_{1,t,\xi} + e_{1,t+1,u}; c_{1,t+1,d} + F_{1,t+1,d}\left(\right) = \theta_{1,t,\xi} + e_{1,t+1}$$

$$c_{2,t+1,u} + F_{2,t+1,u}\left(\right) = \theta_{2,t,\xi}; c_{2,t+1,d} + F_{2,t+1,d}\left(\right) = \theta_{2,t,\xi}$$

"Kernel condition"

$$\frac{\frac{1}{2}\left(c_{1,t+1,u}\right)^{\gamma_1-1} + \frac{1}{2}\left(c_{1,t+1,d}\right)^{\gamma_1-1}}{\left(\omega \times e_{t,\xi}\right)^{\gamma_1-1}} = \frac{\frac{1}{2}\left(c_{2,t+1,u}\right)^{\gamma_2-1} + \frac{1}{2}\left(c_{2,t+1,d}\right)^{\gamma_2-1}}{\left((1-\omega) \times e_{t,\xi}\right)^{\gamma_2-1}}$$

Market clearing

$$\theta_{1,t,\xi} + \theta_{2,t,\xi} = 0$$

- The financial-wealth is calculated as:

$$F_{2,t,\xi} = \frac{\rho}{\left((1-\omega) \times e_{t,\xi}\right)^{\gamma_2-1}} \left\{ \frac{1}{2}\left[\left(c_{2,t+1,u}\right)^{\gamma_2-1} \times \left(c_{2,t+1,u} + F_{2,t+1,u}\right)\right.\right.$$

$$\left.\left. + \frac{1}{2}\left[\left(c_{2,t+1,d}\right)^{\gamma_2-1} \times \left(c_{2,t+1,d} + F_{2,t+1,d}\right)\right]\right\}$$

What the system would have been without the shift

Flow budget constraints of time t

$$c_{1,t} + \theta_{1,t,\xi} \times \frac{1}{1+r} = \theta_{1,t-1}$$

$$c_{2,t} + \theta_{2,t,\xi} \times \frac{1}{1+r} = \theta_{2,t-1}$$

"Kernel condition"

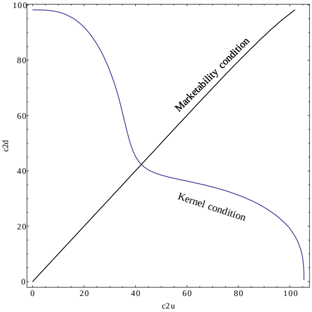
$$\frac{1}{1+r} = \frac{\frac{1}{2}\left(c_{1,t+1,u}\right)^{\gamma_1 - 1} + \frac{1}{2}\left(c_{1,t+1,d}\right)^{\gamma_1 - 1}}{\left(c_{1,t}\right)^{\gamma_1 - 1}}$$

$$= \frac{\frac{1}{2}\left(c_{2,t+1,u}\right)^{\gamma_2 - 1} + \frac{1}{2}\left(c_{2,t+1,d}\right)^{\gamma_2 - 1}}{\left(c_{2,t}\right)^{\gamma_2 - 1}}$$

Market clearing

$$\theta_{1,t,\xi} + \theta_{2,t,\xi} = 0$$

Existence of nodal solution for the Basak-Cuoco example



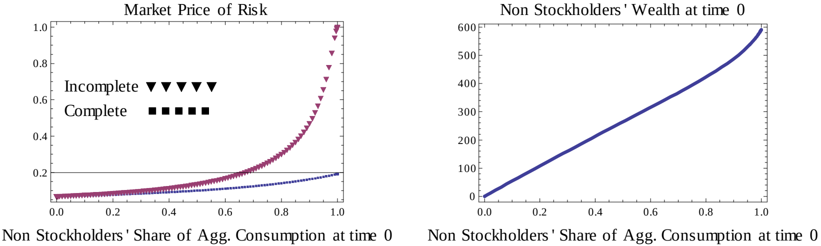
The geometry of the equations system is displayed here. After substituting out $c_{1,t+1,u}$ and $c_{1,t+1,d}$, the system has two remaining equations: the marketability condition and the kernel condition. The picture is calculated for parameter values: $T = 6, \gamma_2 = -5, \gamma_1 = -1, \beta = 0.999, \sigma_\delta = .0357, \mu_\delta = 0.0183$.and for the particular point $\omega = \frac{29}{50}$. The kernel condition alone is affected by the particular choice of ω (the locus shifts down as ω increases). The axes of the picture cover the entire ranges $c_{2,t+1,u} \in [0, e_{t+1,u}]$, $c_{2,t+1,d} \in [0, e_{t+1,d}]$.

Discussion of existence

Making the usual assumptions on convexity, which guarantee existence of equilibrium in complete markets, there are two possible reasons for non existence in incomplete markets

- Drop of rank of endogenous payoff matrix: it has been shown that this occurs on a set of measure zero (Sard's theorem). One says that there is "generic existence" of the equilibrium.
- Initial claim of one person on another is too large to be repaid at equilibrium prices.
- Existence of solution of nodal system: first reason is the only one that is an issue. Proof needed.
- Second reason will be diagnosed at time zero.

Result: equilibrium market price of risk (Sharpe ratio) on the equity market

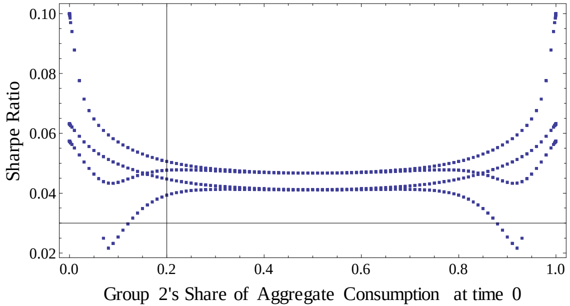


The left-hand panel of this figure shows the market price of risk applicable in the equity market where Group 1 alone "trades". Parameter values are: $T = 6, \gamma_2 = -5, \gamma_1 = -1, \beta = 0.999, \sigma_\delta = .0357, \mu_\delta = 0.0183$. The lower of the two curves, which corresponds to the complete-market case, is provided for comparison. The righthand panel of the figure shows the relationship between time-0 wealth and the time-0 distribution of consumption, which is endogenous to wealth.

Heaton and Lucas (1996)

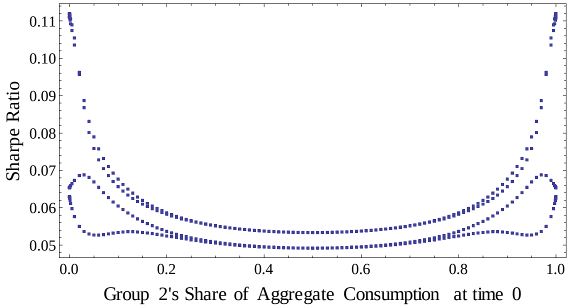
- Model calibrated to real U.S. economy, including idiosyncratic labor shocks observed on panel data
- Two groups of households differ only in the allocation of output to individual labor income
- They have identical risk aversions and discount rates. Because of that, output is only a scale variable, which can be factored out
- Three exogenous state variables describe the exogenous aspects of the economy at any given time:
  - the realized rate of growth of output
  - the share of output paid out as dividend, vs. labor
  - the share of labor income that is paid to Group 1, vs. Group 2
- These follow an eight-state Markov chain, which is calibrated to U.S. data
- One endogenous state variable defined as ω above

Heaton and Lucas (1996): Sharpe ratio in four low-growth states



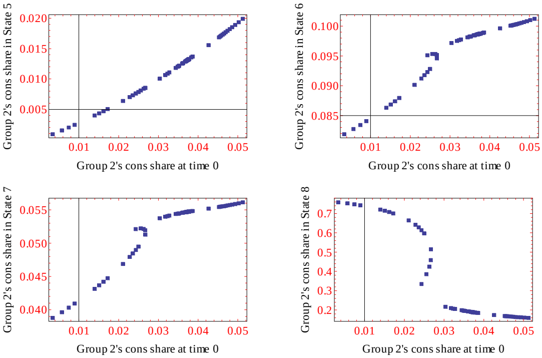
This figure shows the Sharpe ratio on the equity security when the two groups of agents only trade the Bill and the equity, depending on the state of nature the economy is in. This figure contains the four states of nature in which the realized growth rate is low. On the x-axis is the fraction of output consumed by Group 2. Parameter values are as in Heaton and Lucas (1996), Table 2, page 455.

Heaton and Lucas (1996): Sharpe ratio in four high-growth states



This figure shows the Sharpe ratio on the equity security when the two groups of agents only trade the Bill and the equity, depending on the state of nature the economy is in. This figure contains the four states of nature in which the realized growth rate is high. On the x-axis is the fraction of output consumed by Group 2. Parameter values are as in Heaton and Lucas (1996), Table 2, page 455.

Heaton and Lucas (1996): multiple solutions



Consumption correspondence (large scale): This figure shows on the y-axis the consumption share of group 2 in each of the eight states at time 1. On the x-axis is the fraction of output consumed by group 2 at time 0. Parameter values are as in Heaton and Lucas (1996), Table 2, page 455 and $T = 7$, $\gamma = 0.5$.

Future prospects

1. Transactions costs: practically done
2. Default risk: difficult
3. Asymmetric information
4. Large population/ individual risks: approximation only
5. Getting stationary solution directly when there exists a stationary equilibrium
6. Handling autoregressive systems
7. Existence of solution of algebraic system at each node
8. Multiplicity of solutions: correspondences?
9. Good control of numerical issues and speed

Mathematica, MatLab and C codes on my INSEAD webpage:
http://www.insead.edu/facultyresearch/faculty/personal/bdumas/research