Incomplete-Market Equilibria Solved Recursively on an Event Tree*

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ABSTRACT

We develop a method that allows one to compute incomplete-market equilibria routinely for Markovian equilibria (when they exist). The main difficulty that we overcome arises from the set of state variables. There are, of course, exogenous state variables driving the economy but, in an incomplete market, there are also endogenous state variables, which introduce path dependence. We write on an event tree the system of all first-order conditions of all times and states and solve recursively for state prices, which are dual variables. We illustrate this "dual" method and show its many practical advantages by means of several examples.

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> "No information available in period t apart from the level of consumption, c_t, helps predict future consumption, c_{t+1}, in the sense of affecting the expected value of marginal utility. In particular, income or wealth in period t or earlier [is] irrelevant, once c_t is known." Robert E. Hall (1978), corollary 1, page 974.

Because of the large size of non-traded human capital, causing some idiosyncratic risks not to be traded, financial markets in the real world are massively incomplete. For this reason, it is quite possible that the investigation of incomplete-market equilibria will eventually deliver a solution to some of the well-known "puzzles" encountered in financial-market data (excess volatility, equity premium, level of the rate of interest). Missing-market risks should "rock the boat" of traded markets, increasing risk premia and volatility and causing the distribution of wealth in the investor population to act as a dimension of risk that is separate from aggregate wealth.

Several contributions have thrown some light on the issue. Mankiw (1986) and Constantinides and Duffie (1996) show theoretically that risk premia can be increased at will if the variance of idiosyncratic risk is high when the realization of aggregate risk is low while Krueger and Lustig (2010) verify that, absent that relationship, risk premia are not affected by idiosyncratic risk.[1] Telmer (1993), who did not incorporate that relationship, concludes a calibration exercise by saying that "incomplete markets cannot account for the properties of asset returns that are anomalous from the perspective of representative agent theory." Heaton and Lucas (1996) calibrate an equilibrium with two classes of agents, market incompleteness, trading costs and borrowing constraint. They conclude that the borrowing constraint is what makes a difference, more so than the incompleteness. Krusell and Smith (1998) consider a continuum of identical individuals with independent idiosyncratic risks

[1] Levine and Zame (2002) show that incompleteness has little effect if all traders have infinite patience and the same CRRA utility and "the right assets are traded".

and a financial market where only a one-period riskless security is traded. They conclude that the incomplete-market equilibrium is very close to the complete-market equilibrium. Gomes and Michaelides (2008) achieve a high equity premium that is "driven by incomplete risk sharing, borrowing constraints, and a (realistically) calibrated life-cycle earnings profile subject to idiosyncratic shocks." Storesletten, Telmer and Yaron (2007) demonstrate that the net effect of idiosyncratic risk on the Sharpe ratio is small. Basak and Cuoco (1998), however, propose a model with limited participation that shows that, when some people are prevented from accessing a market, the market Sharpe ratio is vastly increased. Guvenen (2009), Guvenen and Kuruscu (2006) show that limited market participation can achieve as good a match of asset-pricing moments as do models of (external) habit formation.

The matter will not be fully settled until the day we have at our disposal a tool to investigate many different case situations. Our goal in the present paper is to develop a method that allows one to compute incomplete-market equilibria routinely for Markovian equilibria (when they exist).[2] "Routinely" means that there would be no need to develop a new trick every time one considers a different economic model. In particular, we would like to be able to process non stationary financial markets.[3] Thirty years after Cox, Ross and Rubinstein (1979) taught us how to calculate the prices of derivatives on an event tree by simple backward induction, we aim to show how a similar formulation can be utilized in computing financial-market equilibria.

[2] The pioneering paper on this question is Marcet and Singleton (1999), previously circulated in 1991.

[3] Many models proposed in Finance lead to non stationary equilibria. For instance, when risk aversions or beliefs differ across investors, the equilibrium is typically non stationary; see Dumas (1989), Kogan *et al.* (2006), Dumas, Kurshev and Uppal (2009). The limited participation equilibrium of Basak and Cuoco (1998), which we study in section III, subsection A, may be non stationary (Hugonnier (2007) characterizes its rapid evolution in continuous time). Sometimes the equilibrium is non stationary only because the economy grows. When utility functions are isoelastic and the growth is geometric, it is possible to re-scale all quantities to make them stationary; see section IV, subsection B.

When considering incomplete-market general equilibrium (GEI) of pure-exchange economies, the main difficulty to be overcome arises from the set of state variables. There are, of course, exogenous state variables driving the economy (for instance, output) but, in an incomplete market, there are also endogenous state variables (say, to fix ideas temporarily, "the distribution of wealth" in the population). Mathematicians say that the system is "forward-backward" in time: exogenous state variables are subject to an initial condition while wealth is subject to both an initial and a terminal condition.

We start from the "traditional formulation" based on dynamic programming but we introduce several changes. First, we use for endogenous state variables not the acquired wealths of the agents but their individual state prices or, equivalently, their current consumptions relative to each other. Second, we regroup the agents' first-order conditions so that, at any given time and in any given node, we do not solve simultaneously for current portfolios and current consumption but solve, instead, simultaneously for the current portfolio and the *agents' state prices (or consumptions) in all nodes that succeed the current one.* Since today's portfolio directly finances tomorrow's consumption, the technique allows some amount of decoupling between time periods and avoids the need for an "expectations step" of the kind that is present in many algorithms used by macroeconomists and which captures the way in which agents forecast the future.[4] In our method, the future is part of the current solution. *The result will be a recursive construction of tomorrow's individual agents' state prices (or consumptions) as functions of today's state prices (or consumptions).* Third, we do not carry backward and interpolate the value functions of agents' dynamic programs. Instead, we carry backwards agents' wealths and securities' prices as functions of the endogenous state variables. The wealths in question are not the wealths carried forward by

[4] See below, in section IV, our brief description of the algorithm implemented by Krusell and Smith (1998), for instance. See also the concept of "expectations correspondence" in Duffie *et al.* (1994).

agents, which do not enter the calculation at all. They are, instead, the present values of future net expenditures, or the wealths needed so that agents can continue with their optimal program at the current and future prices of securities. In this way, we never have to worry about whether the chosen strategies can actually be financed by future endowments and accumulated wealth. When time 0 is reached, we obtain the incomplete-market analog of a Negishi map.

The technique brings several key benefits over the traditional dynamic-program with tatonnement approach: (i) *Wealth in this construction is not a state variable*, a property which brings the major advantage that we never have to limit the positions taken by agents and we do not have to limit endogenously the domain of agents' wealth state variables, to guarantee that each of them has enough wealth remaining to continue trading, (ii) For this reason, the system, which was originally forward-backward becomes entirely backward all the way to time 0, which is the only point at which we have to make sure, by adjusting the initial value of the endogenous state variables, that the present value of future net expenditures jibes with the initial claims of each agent, which are givens of the problem, (iii) In contrast to the traditional dynamic programming approach, the algorithm is not limited to a relatively small number of assets. That number only increases the size of the equation system to be solved at each node. The main dimensional limitation of the algorithm, which would be a limitation of the traditional approach anyway, is actually the number of agents in the economy, (iv) While, in traditional dynamic programing, derivatives of the value function appear in first-order condition, here we never have to take a derivative of a function that has been interpolated. As is well-known in numerical analysis, the derivative of an approximate function is typically not a good approximation of the derivative of that function.

It is probably impossible to do a proper review of the relevant literature without writing a book. We can at best crisscross it. There exist two classes of methods to compute a GEI solution: the global and the recursive ones. In the "global" method, first-order conditions and market-clearing conditions at all nodes are stacked into one vast system of nonlinear equations, to which the initial ($t = 0$) conditions are appended. The system is solved as an algebraic system, i.e., as a system where the unknowns are the numbers describing the entire time path (consumption of each agent, portfolio holdings and asset prices) of the economy until a fixed termination date. The disadvantage of this approach is that the number of equations and variables grows exponentially in the horizon T and thus methods of this type have only been applied to models with a very short horizon. Cuoco and He (2001) propose to write such a system on a binomial tree.

The recursive (or backward-induction) algorithm is a bit more delicate to define. Loosely speaking, it consists in figuring out the equilibrium at time t once the equilibrium at time $t + 1$ has already been calculated. At all times, the solution is calculated not as a set of numbers but as a set of functions of the state of the economy. The exact implementation depends on the choice of a parameterization, or state description of the economy at each point in time. As noted, in GEI models, there are two kinds of state variables: exogenous and endogenous ones. While our approach is recursive, we differ from the "traditional" recursive method, which has been used mostly to derive stationary equilibria,[5] in our choice of the endogenous state variables. Cuoco and He (1994) propose a related recursive method in a continuous-time setting.[6]

[5]Ljungqvist and Sargent (2000), chapter 17, propose recursive methods to solve for infinite-horizon stationary incomplete-market equilibria. They give an account of the work of Bewley (1987), Huggett (1993) and Aiyagari (1994).

[6]In the present paper, we prefer to stay away from continuous time for two reasons. First, the infinite dimension of spaces opens possibilities for non existence of equilibria and for the presence of a type of "bubbles" that do not arise in a finite-dimensional space (See Heston, Lowenstein and Willard (2007) and Hugonnier

We tend to prefer a recursive approach as being less likely to go haywire numerically than a global approach. Further, when the exogenous tree can be made recombining, thereby drastically cutting down the number of nodes, the recursive method delivers a side benefit, which is of huge practical importance, as compared to the global one. We discuss in section IV, subsection A the relative merits of the global *vs.* the recursive approaches.

Our method is related to two sets of contributions from the field of Mathematical Economics. Papers of the first set are those that demonstrate the generic existence of equilibrium in an incomplete-market stochastic finance economy in which long-lived real assets are traded.[7] They rely on a concept variously called "pseudo-equilibrium" or "no-arbitrage equilibrium", which involved state prices as unknowns. We use also a definition of equilibrium in which the unknowns are state prices. The second set of papers pertains to the existence of a recursive formulation of the equilibrium when the exogenous state variables are Markovian and a stationary equilibrium is sought. They discuss the choice of the endogenous state variables that permit recursivity. Kubler and Schmedders (2002), in particular, provide examples showing that the distribution of wealth in the population, and even the equilibrium asset holdings of investors do not constitute a sufficient state space. Here, we shall illustrate that the distribution of individual-specific state prices or, equivalently, the distribution of consumption is a natural choice of endogenous state variables.

Optimizing without a Bellman value function has been done before in Applied Mathematics, in Finance and in Economics. Pontryagin (1962) invented the Maximum Principle

(2007)). Here, we only consider finite-horizon economies in discrete time so that both the state space and the time space are finite. For that reason, we have no need to place constraints on wealth and/or borrowing to avoid Ponzi schemes. Secondly, continuous-time models require the solution of partial differential equations, approximated by means of finite-differences, over an artificially bounded domain. These involve boundary conditions on the edges of the domain that are difficult to establish *a priori*.

[7]See Cass (2006, first distributed in 1984), Duffie and Shafer (1985, 1986), Magill and Shafer (1990), Husseini, Lasry and Magill (1990), Hirsch, Magill and Shafer (1990), Magill and Shafer (1991) and the textbook Magill and Quinzii (1996).

of optimal control theory precisely in order to handle situations where the value function may not be available in a differentiable form. The Maximum Principle was generalized to stochastic settings in Bismut (1973).[8] According to the stochastic Maximum Principle, the conditions of optimality form, in continuous time, a system of forward and backward stochastic differential equations (BSDE) of the type analyzed by Pardoux and Peng (1990) and El Karoui, Peng and Quenez (1997). For that reason, it is sometimes referred to as "the BSDE approach". In the current paper, we exhibit a way to solve the optimization problem in a purely backward fashion directly at market equilibrium. In Finance, Cox and Huang (1989) for complete markets and He and Pearson (1991) and Cvitanic and Karatzas (1992) for incomplete ones have developed the "dual" or "martingale" approach to optimization of portfolios. Our choice of state variables is akin to that approach and Cuoco and He (1994) also draws from it.[9] In Economics, the term "policy-function iteration" refers to a method in which future decisions of agents are calculated directly as functions of today's decisions without the medium of a value function. Wright and Williams (1982a, 1982b, 1984) have used it to analyze stationary partial equilibria of the commodities market.[10] Lucas and Stokey (1987) and Coleman (1990, 1991) have used it to analyze stationary general equilibria.

The balance of the paper is organized as follows. In section I, we write the first-order conditions that must prevail at each node of the tree. In section II, we explain how the solution of the intertemporal system can be obtained recursively. Section III contains three canonical examples of the application of our method and section IV three additional examples, one of

[8] As can be gathered from Bismut (1975), we could have reached our equation system (4) below without using dynamic programming and without invoking a value function, writing instead the evolution of the costate or dual variable $\phi_{l,t}$.

[9] For an application to decisions on a tree, see Detemple and Sundaresan (1999). Bizid and Jouin (2001, 2005) have established bounds on equilibrium prices of securities in an incomplete market, using a similar martingale approach.

[10] Their demand functions for commodities are postulated.

which is calibrated to the U.S. economy. We use these three examples also to compare our own algorithm to two others. In section V, we discuss a problematic situation. The final section concludes with some prospective developments.

I. The first-order conditions at a node

A. The economy

Time is discrete, from 0 to T. We start with an event tree $(\Omega, \mathcal{F})$ where Ω is a sample set and $\mathcal{F} \equiv \{\mathbb{F}_t\}_{t=0}^T$ a filtration ($\mathbb{F}_T = \Omega$). We will interpret the filtration $\mathcal{F}$ as a tree structure in the usual way, i.e., by identifying the sets from each partition $\mathbb{F}_t$ as the "time-t nodes" on the tree. We follow the standard definition of an event tree, as in Magill-Quinzii (1996), Section 4.18. The unique predecessor of node ξ is denoted ξ^-. The set of nodes that at time $t+1$ succeed node ξ at time t is called $\xi^+ \subset \mathbb{F}_{t+1}$ and a generic successor of node ξ is typically denoted $\eta : \eta \in \xi^+$. These are K_ξ in number. The state-space Ω is endowed with a probability measure $\pi(\sigma) \in]0,1], \quad \sigma \in \Omega, \quad \sum_{\sigma \in \Omega} \pi(\sigma) = 1$.

The financial market is populated with L investors indexed by $l = 1, \ldots, L$ who receive a set of exogenous time×state sequences of individual endowments $\{e_t \in \mathbb{R}^L_{++}; 0 \leq t \leq T\}$ adapted to $(\Omega, \mathcal{F})$. For the purpose of our recursive method, it is sufficient for the filtration of the event tree to be generated by the exogenous state variables e. Because the tree only involves the exogenous endowments, it can in principle be chosen to be made *recombining* when the endowments are Markovian, which is a great computational advantage compared to the global-solution approach (see section IV, subsection A below for a comparison). Investors may also be endowed with initial claims on each other: $W_0 \in \mathbb{R}^L$, $\sum_{l=1}^L W_{l,0} = 0$.

At each point in time, the agents must consume some strictly positive amounts $c_{l,t} > 0$ of a single perishable good, which we use as the numeraire in the economy. As in Debreu (1970, 1972) and Duffie and Shafer (1986), we make the smooth-preference assumption, i.e., the consumption preferences of the agents are expressed in terms of the utility functions $U_{l,t,\xi} : \mathbb{R}_{++} \mapsto \mathbb{R}$ which are assumed increasing, twice continuously differentiable and strictly concave. The goal of agent l at time t in node $\xi \in \mathbb{F}_t$ is to maximize the quantity

$$J_{l,t,\xi}(c_l) \triangleq U_{l,t,\xi}\left(c_{l,t,\xi}\right) + \sum_{\tau=1}^{T-t} \mathbb{E}_{t,\xi}\left[U_{l,t+\tau}\left(c_{l,t+\tau}\right)\right],$$

In order to reduce the number of subscripts, we keep the reference to node ξ of time t implicit and, for instance, write the above objective function simply as:

$$J_{l,t}(c_l) \triangleq U_{l,t}\left(c_{l,t}\right) + \sum_{\tau=1}^{T-t} \mathbb{E}_{t}\left[U_{l,t+\tau}\left(c_{l,t+\tau}\right)\right],$$

We make an assumption on utility such that the agents choose strictly positive consumption, if that is at all feasible (i.e., as long as their budget set is not empty). That is an Inada assumption: $\lim_{x \to 0} U'_{l,t}(x) = +\infty$.

In the financial market, there are $N \geq 1$ securities defined by their payoffs or "dividends" $\left\{\delta_t \in \mathbb{R}^N\right\}$. The market is incomplete in the sense that $N < K_\xi$ for at least some node ξ. In some examples, investors hold the securities long and some other investors hold them short as they are "in zero net supply". In some other examples, the securities' payoffs may include some of the endowments, in which case we say that the securities are in positive net supply. One is just an accounting transformation of the other. We develop the equation system under the first instance but we remain free to present later some examples under the second instance. The prices of the securities are denoted: $\{S_{t,n}; 1 \leq n \leq N; 0 \leq t \leq T\}$. We impose: $S_T \equiv 0$.

Any portion of the investor's wealth that is not consumed at any time t is invested in a portfolio of securities described by the vector $\theta_{l,t} \in \mathbb{R}^N$, which represents the numbers of shares held. The *entering* wealth for time t, not including the endowment $e_{l,t}$ received at t, is defined thus:

$$W_{l,t} \triangleq \theta_{l,t-1} \cdot (S_t + \delta_t).$$

Investor l's budget set for the entering wealth level w at time t is:

$$\mathbb{B}_{l,t}(w) \triangleq \left\{ c_l \text{ adapted to } \mathcal{F} \;\middle|\; \begin{array}{c} c_{l,t+\tau} + \theta_{l,t+\tau} \cdot S_{t+\tau} = e_{l,t+\tau} + W_{l,t+\tau}, \\ \tau = 0, \ldots, T-t \\ W_{l,t} = w \\ \theta_{l,T} \equiv 0 \\ \theta_l \text{ adapted to } \mathcal{F} \end{array} \right\} \tag{1}$$

Investor l's *value function* for time t is, therefore, given by:[11]

$$V_{l,t}(w) \triangleq \sup\left\{J_{l,t}(c_l); c_l \in \mathbb{B}_{l,t}(w)\right\}, \tag{2}$$

It is verified in an appendix available on request, that the Principle of Dynamic Programming applies, i.e., the goal $J_{l,0}(c_l)$ of investor l at time $t = 0$ is maximized if and only if his goal $J_{l,t}(c_l)$ at all times and in all possible states of the economy is maximized. Consequently, as illustrated in figure 1, when investor l is faced with entering wealth $W_{l,t}$, local price vector S_t, and new endowment $e_{l,t}$, he computes his immediate consumption $c_{l,t}$, his immediate

[11] We set $V_t^i(W_t^i) = -\infty$ if the budget set is empty.

trading strategy $\theta_{l,t}$ and his local (in time and state of the economy) *Arrow-Debreu shadow prices* $\phi_{l,t} \in \mathbb{R}$ in such a way that the following first-order conditions are satisfied, with the exception of the third one when $t = T$:

$$\begin{aligned} U'_{l,t}(c_{l,t}) &= \phi_{l,t}, \\ c_{l,t} + \theta_{l,t} \cdot S_t &= e_{l,t} + W_{l,t}, \\ \mathbb{E}_t\left[V'_{l,t+1}\left[\theta_{l,t} \cdot (S_{t+1} + \delta_{t+1})\right] \times (S_{n,t+1} + \delta_{n,t+1})\right] &= \phi_{l,t} S_{n,t}\,, \quad 1 \leq n \leq N. \end{aligned} \tag{3}$$

B. The dual first-order conditions

We now endeavor to remove the value function from the problem. A straightforward application of the envelope theorem gives

$$V'_{l,t}(W_{l,t}) = \phi_{l,t}(W_{l,t}). \tag{4}$$

We substitute equation (4) into (3) and show that the resulting first-order conditions are necessary and sufficient for optimality, which is the main result on which our method rests:

THEOREM 1: *Given a price process S and initial wealths $W_{l,0}$, the choice of consumption plans c_l, trading strategies θ_l and state prices ϕ_l maximizes investor l's goal at all times and in all possible states of the economy if and only if the following three conditions, except for the third one when $t = T$, are satisfied for any $0 \leq t \leq T$ and in any state $\xi \in \mathbb{F}_t$:*

$$\begin{aligned} U'_{l,t,}(c_{l,t}) &= \phi_{l,t}, \\ c_{l,t} + \theta_{l,t} \cdot S_t &= e_{l,t} + W_{l,t}, \\ \mathbb{E}_t\left[\phi_{l,t+1} \times (S_{n,t+1} + \delta_{n,t+1})\right] &= \phi_{l,t} S_{n,t}, 1 \leq n \leq N\,. \end{aligned} \tag{5}$$

Furthermore, the value functions $V_{l,t}$, treated as functions of the entering wealths $W_{l,t}$ for time t, are concave in any state $\xi \in \mathbb{F}_t$, for any $0 \leq t \leq T$, and, if it exists, the solution (c_l, ϕ_l) is necessarily unique.

The proof (in the appendix) amounts to showing by backward induction that the function $V_{l,t}$ is concave.

C. The equilibrium

A financial-market equilibrium is defined in, e.g., Magill-Quinzii (1996), page 228, as a set of securities prices, portfolios and consumption allocations in the population such that the securities' markets clear: $\sum_{l=1}^{L} \theta_{l,t} = 0$. The issue of the existence of a financial-market equilibrium in an incomplete financial market, when securities are long lived (which means that they are not just next–time payoff securities) has been the subject of several papers.[12] They have found that, under the set of assumptions that we make in the present paper, equilibrium can fail to exist in the economy described by:

$$\Omega,\ \mathcal{F},\ \pi,\ W_{l,0} = 0,\ e_l,\ \delta_n,\ U_{l,t};\quad 1 \leq l \leq L,\quad 1 \leq n \leq N \tag{6}$$

only on a set of points of the dataset $\{e_l, \delta_n;\ 1 \leq l \leq L,\ 1 \leq n \leq N\}$ of zero measure.[13] This is called "generic existence". Equilibrium fails to exist if it so happens that the matrix of

[12] See the references already mentioned in the introduction: Cass (1984), Duffie and Shafer (1985, 1986), Magill and Shafer (1990), Husseini, Lasry and Magill (1990), Hirsch, Magill and Shafer (1990), Magill and Shafer (1991) and the textbook Magill and Quinzii (1996).

[13] This result was established for the case $W_{l,0} = 0$. In applications where initial claims at time 0 are not zero (but, of course, sum to zero across the population), the sizes of the claims must be below some upper bound. That issue is addressed below.

one-period payoffs inclusive of capital gains, in some state of the economy, fails to be of full rank N.

Definition 1: *The choice of a price process* S*, consumption plans* c_l*, trading strategies* θ_l *and state prices* ϕ_l*, for* $1 \leq l \leq L$*, is an equilibrium for the economy (6), if all conditions in (5) are satisfied – i.e., with this choice all agents maximize their goals under the price process* S *at all times and in all states of the economy – and, in addition, the following market-clearing condition holds for any* $0 \leq t \leq T$*, any security* n *and in any state* $\xi \in \mathbb{F}_t$

$$\sum_{l=1}^{L} \theta_{n,l,t} = 0. \tag{7}$$

Thus, in order to obtain an equilibrium, one must solve the system made of (5) (for $l = 1, .., L$) and (7) – for all times and all states of the economy.

II. Recursivity

We can treat (5) and (7) as a system of conditions grouped by points in time for $t = 0, 1, \ldots, T$, in which case, at time t, one must be able to compute the prices at which securities are to be traded as well as the time-t consumption levels for all agents, which appear in the first and second equations of (5). However, it would be hard to solve the system recursively in the backward way because the unknowns at time t include consumptions at time t, $c_{l,t}$, whereas the third component of equations (5) if rewritten as:

$$\mathbb{E}_t \left[U'_{l,t+1} \left(c_{l,t+1} \right) \times \left(S_{n,t+1} + \delta_{n,t+1} \right) \right] = \phi_{l,t} S_{n,t}$$

can be seen to be a *restriction* on consumptions at time $t+1$, which at time t would already be solved for. In this form, the system is simultaneously forward and backward at each point in time. It is for this reason that den Haan (1997), Krusell and Smith (1998), Storesletten, Telmer and Yaron (2006) and Gomes and Michaelides (2008) include forecasting functions in their equation system.[14] They must iterate backward and forward within a set of basis functions in order to converge on self-fulfilling forecasting functions. In what follows, we propose a method that completely obviates the need to include forecasting functions. In our method, the forecasts are part of the solution at each node.

Another observation motivates the re-formulation that we introduce below. When wealth is an endogenous state variable, as is the case so far, it is hard to decide *a priori* what should be its domain, over which the various policy functions would be defined and interpolated. The domain is endogenous. If an investor has excessively negative wealth, calculated at the endogenous securities prices, no equilibrium exists as it becomes impossible for him/her later to repay.[15] One would be able to determine the domain at each point in time only after the algorithm has reached time 0, which is the point at which the initial wealth conditions are specified. To deal with this problem, most researchers take an iterative approach in which the distribution of wealth is fixed on some interval, the model is then solved numerically to examine whether the limits on wealth are binding. If the limits bind, the limits are relaxed, the model is solved once more till one reaches the "natural borrowing limit".[16]

[14] The method is often referred to as "the parameterized expectations approach" (PEA).

[15] Compare with Schmedders, Judd and Kubler (2002).

[16] See Aiyagari (1994).

A. The main result

In order to make a recursive solution possible, we re-formulate the system so that it can be solved backward all the way from time T to time 0, going backward only once. The reformulation involves two separate modifications.

First, we re-cast the flow-budget condition (the second condition in (5)) in terms of agent l's wealth *exiting* time t, which is defined as $F_{l,t} \triangleq \theta_{l,t} \cdot S_t$. As the wealth entering time t is $W_{l,t} = \theta_{l,t-1} \cdot (S_t + \delta_t)$, the flow-budget constraint at time t can be written equivalently as:

$$c_{l,t} + F_{l,t} = e_{l,t} + \theta_{l,t-1} \cdot (S_t + \delta_t) \tag{8}$$

Secondly, we introduce a crucial time-shift, or regrouping, which makes the solution by backward induction possible. To be precise, we combine the first equation set of (5), the second equation set of (5) rewritten as (8) and equation (7), all *written for time* $t+1$, with the third equation of (5) written for time t. Consequently, we associate with time $0 \leq t \leq T-1$

and with state $\xi \in \mathbb{F}_t$ the following set of nodal conditions:[17]

$$
\begin{aligned}
U'_{l,t+1}\left(c_{l,t+1,\eta}\right) &= \phi_{l,t+1,\eta}, \quad 1 \leq l \leq L,\ \eta \in \xi^{+}\,, \\
c_{l,t+1,\eta} + F_{l,t+1,\eta} &= e_{l,t+1,\eta} + \theta_{l,t} \cdot \left(S_{t+1,\eta} + \delta_{t+1,\eta}\right), \\
&\quad 1 \leq l \leq L,\ \eta \in \xi^{+} \\
\frac{\mathbb{E}_t\left[\phi_{l,t+1} \times \left(S_{n,t+1} + \delta_{n,t+1}\right)\right]}{\phi_{l,t}} &= \frac{\mathbb{E}_t\left[\phi_{L,t+1} \times \left(S_{n,t+1} + \delta_{n,t+1}\right)\right]}{\phi_{L,t}}\,, \\
&\quad 1 \leq n \leq N,\ 1 \leq l \leq L-1\,, \\
\sum_{l=1}^{L} \theta_{n,l,t} &= 0; 1 \leq n \leq N
\end{aligned}
\tag{9}
$$

For any given node $\xi \in \mathbb{F}_t$, these conditions must hold simultaneously across its immediate successors $\eta \in \xi^{+}$. In other words, as illustrated in figure 2, we consider artificially that the decisions to be made at time t are the portfolio decision at time t *and the consumption decisions at time* $t+1$, instead of time t.

System (9) contains four subsets of equations: The first subset provides the link between consumption and state prices. The second subset is the flow budget constraint for the states of time $t+1$. It could also be called “the marketability condition” because it imposes that, in this incomplete market, there exist a portfolio θ_t chosen at time t that makes the consumption-wealth plan of time $t+1$ feasible. The third subset says that all investors must agree on the prices of traded assets. We call it “the kernel condition” because it restricts the state prices $\phi_{l,t+1,\eta}$ to lie in some linear subspace. Finally, the fourth subset is the market-clearing condition.

[17] Once we break the global system into local systems of equations, one for each node, the local equilibrium, which is defined for given future price and wealth functions, can be viewed as a “temporary competitive equilibrium” in the sense of Grandmont (1977).

The unknowns are $\{c_{l,t+1,\eta}, \phi_{l,t+1,\eta};\ 1 \leq l \leq L,\ 1 \leq \eta \leq K_t\}$ and $\{\theta_{l,t};\ 1 \leq l \leq L,\ 1 \leq n \leq N\}$. Since the equation system is linear in the portfolio choice θ_l and since that choice is unconstrained, θ_l can be eliminated from the equation system, reducing the number of unknowns and the number of equations by $N \times L$.[18] Besides the exogenous endowments $e_{l,t+1,\eta}$, the "givens" are:

- the individual state prices of time t, $\left\{\phi_{l,t}\right\}_{l=1}^{L}$, which must be treated as endogenous state variables,

- the future securities' prices $S_{n,t+1,\eta}$, which have been obtained point by point by the backward induction formula:

$$S_{n,t+1,\eta} = \frac{\mathbb{E}_{t+1,\eta}\left[\phi_{l,t+2} \times (S_{n,t+2} + \delta_{n,t+2})\right]}{\phi_{l,t+1,\eta}}\ ,\ S_{n,T} \equiv 0\ ,\ 1 \leq n \leq N\ ,\ 1 \leq l \leq L\ , \quad (10)$$

  and interpolated so that they appear in the system as *functions* $S_{n,t+1,\eta}\left(\left\{\phi_{l,t+1,\eta}\right\}_{l=1}^{L}\right)$,

- and finally the future investors' exiting wealths $F_{l,t+1,\eta}$, which have also been obtained point by point by backward induction:

$$F_{l,t+1,\eta} = \frac{\mathbb{E}_{t+1,\eta}\left[\phi_{l,t+2} \times (F_{l,t+2} + c_{l,t+2} - e_{l,t+2})\right]}{\phi_{l,t+1,\eta}}\ ,\ F_{l,T} \equiv 0\ ,\ 1 \leq l \leq L\ , \quad (11)$$

  and interpolated so that they appear in the system as *functions* $F_{l,t+1,\eta}\left(\left\{\phi_{l,t+1,\eta}\right\}_{l=1}^{L}\right)$. The last equation follows from (10) by dot multiplying by $\theta_{l,t+1,\eta}$ and invoking the second equation of (9).

[18] If the market were complete, i.e., $N = K_t$ for all t and ξ, this elimination would be sufficient for all flow budget constraints in (9) to disappear, leading to a well-known separation between consumption decisions and portfolio decisions. Such is not the case in an incomplete market.

It is clear that (9) exhausts all conditions defining equilibrium, i.e., (5) and (7), except for the first two conditions in (5) and condition (7) at $t = 0$, which together are the only "forward" conditions remaining and which we can write as:

$$\begin{aligned} U'_{l,0}(c_{l,0}) = \phi_{l,0}\,, \quad 1 \leq l \leq L, \\ c_{l,0} + \theta_{l,0} \cdot S_0 = e_{l,0} + W_{l,0}\,, \quad 1 \leq l \leq L\,, \\ \sum_{l=1}^{L} \theta_{n,l,0} = 0; 1 \leq n \leq N \end{aligned} \tag{12}$$

Exiting wealth is calculated backward all the way till time 0, as the present value of future net expenditures (see (11)). It should be interpreted as the wealth needed by each investor in order for him or her to be able to carry on his/her consumption program. The wealth actually owned (entering wealth W) does not enter the algorithm except at the very end, once time 0 is reached. At that time, we obtain a "*Negishi map*" mapping the state prices into required wealth $\theta_{l,0} \cdot S_0$.[19] The map is a very useful tool. For the given level of initial wealth W_0, we use the Negishi map to solve for the initial allocation of consumption, which will then by forward propagation provide all the values of the variables at all the nodes. The image of the Negishi mapping is typically a bounded set of values of wealth. If so and if the initial wealth net of time-zero excess consumption, $e_{l,0} + W_{l,0} - c_{l,0}$, falls within the image of the mapping, there exists an equilibrium. Otherwise not. If the Negishi map is monotonic, the equilibrium is unique.

[19] An example of a Negishi map is provided below as figure 4, bottom panel.

Because all investors agree on traded securities prices (as per the kernel restrictions), the recursions (10) can equivalently be written on the basis of a single investor's state prices:

$$
\begin{aligned}
S_{n,t+1,\eta} &= \frac{\mathbb{E}_{t+1,\eta}\left[\phi_{L,t+2} \times (S_{n,t+2} + \delta_{n,t+2})\right]}{\phi_{L,t+1,\eta}}; \\
S_{n,T} &\equiv 0\,, \quad 1 \leq n \leq N
\end{aligned}
\tag{13}
$$

and the recursion (11) can equivalently be written:

$$
F_{l,t+1,\eta} = \frac{\mathbb{E}_{t+1,\eta}\left[\phi_{L,t+2} \times (F_{l,t+2} + c_{l,t+2} - e_{l,t+2})\right]}{\phi_{L,t+1,\eta}}\,,\ F_{l,T} \equiv 0\,,\ 1 \leq l \leq L\,. \tag{14}
$$

We have now formulated the system in a backward state–price recursive way, in the following sense of the term:

Definition 2: *A state–price recursive equilibrium for an economy is a set of functions* $S_{n,t}\left(\{\phi_l\}_{l=1}^L\right)$ *and* $F_{l,t}\left(\{\phi_l\}_{l=1}^L\right)$ *defined over* $\phi_1 \in \left]U_1'\left(\sum_{l=1}^L e_{l,t}\right), +\infty\right[$, $\phi_2 \in \left]U_2'\left(\sum_{l=1}^L e_{l,t} - U_1'^{-1}(\phi_1)\right), +\infty\right[$ *etc., such that equations (9), (13) and (14) are satisfied.*

The equilibria we have constructed are recursive in the sense that there exist state variables (namely time, the exogenous node of the tree as exogenous variables and current state prices as endogenous variables) such that all prices and decisions can be expressed time after time as functions of these state variables. The emergence of state prices as natural endogenous state variables is very much in line with the stochastic Maximum Principle of Bismut (1973) where the co-state variables, and generally all dual state variables play a central role in orchestrating the choice of the optimal path(s). Unfortunately, we are not in a position to assert that, given a Markovian process for exogenous state variables, the equilibrium process, where equilibrium is as defined above, is Markovian with respect to this extended set of state variables so that there would exist a state-price recursive solution.

Quite obviously, if the definition of equilibrium were changed, to include, say, constraints on investors' choices, additional dual variables would be needed, such as the multipliers of the constraints.

A more restrictive concept of recursive equilibrium has been defined by Kubler and Schmedders (2002) in the context of stationary equilibria. There, agents have an infinite horizon and the decision and price functions in a recursive equilibrium are required to be independent of time. A stationary equilibrium may or may not be the limit of a finite-horizon equilibrium as one takes the horizon date to infinity. Our algorithm is not meant to calculate stationary equilibria and we make no claim that, even after a large number of periods, it would find the stationary equilibria discussed in Kubler and Schmedders.

The system has a homogeneity property, which has been noted before (for instance, by Cuoco and He (1994)) and which involves the current values of the endogenous state variables $\phi_{l,t}$, $1 \leq l \leq L$. These appear only in the kernel condition and it is obvious by inspection that only the ratios $\phi_{l,t}/\phi_{L,t}$, $1 \leq l \leq L-1$ matter and not the levels of these variables. The solution of the system is homogeneous of degree 0 in $\left\{\phi_{l,t};\ 1 \leq l \leq L\right\}$. To carry out a calculation, therefore, the natural endogenous state variables are $\left\{\phi_{l,t}/\sum_{l'=1}^{L}\phi_{l',t};\ 1 \leq l \leq L-1\right\}$.

B. A matter of great numerical convenience

However, variables that are values of *any* other one-to-one function of the ratios $\left\{\phi_{l,t}/\sum_{l'=1}^{L}\phi_{l',t}\right\}$ will do as well. We now choose one such function that will simplify the numerics. Define total endowment: $e_t \triangleq \sum_{l=1}^{L} e_{l,t}$ and the current share of consumption of agent l:

$$\omega_{l,t} \triangleq \frac{c_{l,t}}{e_t}, \quad 1 \leq l \leq L-1\,.$$

Given the monotonicity of marginal utility, at any given node (t,ξ) the ratios $\phi_{l,t}/\sum_{l'=1}^{L}\phi_{l',t}$, which can be written equivalently as $U'_{l,t}(c_{l,t})/\sum_{l'=1}^{L}U'_{l,t}(c_{l,t})$, are in a one-to-one relation with the quantities $\omega_{l,t}$. Hence, we can use $\{\omega_{l,t}, 1 \leq l \leq L-1\}$ as our endogenous state variables.[20]

In the system (9), substitute out the state prices by means of the first-order conditions for consumption choice and introduce thereby the current shares of consumption $\omega_{l,t}$:

Flow budget constraint or "marketability" condition:

$$c_{l,t+1,\eta} + F_{l,t+1,\eta} = e_{l,t+1,\eta} + \theta_{l,t} \cdot (S_{t+1,\eta} + \delta_{t+1,\eta})\,,\ 1 \leq l \leq L\,,\ \eta \in \xi^+\,,$$

"Kernel" condition:

$$\frac{\mathbb{E}_t\left[U'_{l,t+1}(c_{l,t+1}) \times (S_{n,t+1} + \delta_{n,t+1})\right]}{U'_{l,t}(\omega_{l,t} \times e_t)} = \frac{\mathbb{E}_t\left[U'_{L,t+1}(c_{L,t+1}) \times (S_{n,t+1} + \delta_{n,t+1})\right]}{U'_{L,t}\left(\left(1 - \sum_{l=1}^{L-1}\omega_{l,t}\right) \times e_t\right)}\,, \qquad (15)$$

$$1 \leq n \leq N\,,\ 1 \leq l \leq L-1\,,$$

Market clearing:

$$\sum_{l=1}^{L}\theta_{n,l,t} = 0; 1 \leq n \leq N$$

In this final form, at any given current node, the solution amounts to calculating future consumptions $\left\{c_{l,t+1,\eta};\ 1 \leq l \leq L,\ \eta \in \xi^+\right\}$ simultaneously in all the successor nodes, for each value of the distribution of consumption at the current node $\{\omega_{l,t};\ 1 \leq l \leq L-1\}$. The distribution of consumption in the population is our choice of endogenous state variable, which achieves recursivity. We conjecture that the solution of this system always exists but

[20] Chien, Cole and Lustig (2008, appendix) also uses that transformation.

we have no proof as yet. Below (section III, subsection A), we illustrate on an example the geometry of this system.

The functions $S_{n,t}$ and $F_{l,t}$ to be carried backward are themselves homogeneous of degree 0 in $\left\{\phi_{l,t}; 1 \leq l \leq L\right\}$ and can be expressed as functions of the variables $\{\omega_{l,t};\ 1 \leq l \leq L-1\}$: $F_{l,t} = F_{l,t}\left(\{\omega_{l,t}\}_{l=1}^{L-1}\right)$, $S_{n,t} = S_{n,t}\left(\{\omega_{l,t}\}_{l=1}^{L-1}\right)$. The great numerical benefit of this choice of variables is that all variables and function values remain bounded and continuous over the entire simplex $\left\{\omega_{l,t} > 0; \sum_{l=1}^{L} \omega_{l,t} = 1\right\}$. Intuitively, this follows from the fact that the distribution of consumption at date $t+1$ is often not very far from the distribution of consumption at date t.

After the system (15) is solved at time t at node ξ, the functions corresponding to that node are calculated point by point by the time-t analogs of formulae (13) and (14), and interpolated over $\left\{\omega_{l,t} > 0; \sum_{l=1}^{L} \omega_{l,t} = 1\right\}$. We have now formulated the system in a consumption-recursive way.

III. Examples of the Basak-and-Cuoco variety

A. The Basak-and-Cuoco (1998) equilibrium

Our first example[21] application is the simplest one, because it requires only the backward induction of one function $F_{l,t}$. No functions $S_{n,t}$ for asset prices need to be carried backward.

We consider an economy in which there are two groups of agents. Agents of group 1 receive an endowment stream e which follows a geometric Brownian motion. We capture that endowment with a re-combining binomial tree with fixed drift and volatility as is done in Cox, Ross and Rubinstein (1979).[22] We set the transition probabilities π at $\frac{1}{2}$. Agents of

[21] The Mathematica, MatLab and C codes for this example are available from the authors' websites.
[22] Or more precisely in Jarrow and Rudd (1983).

group 2 receive no endowment stream but they are able to consume because they start their lives with some initial financial claims on agents of group 1:

$$
\begin{aligned}
W_{2,0} &> 0 \\
W_{1,0} &= -W_{2,0}
\end{aligned}
$$

The market is incomplete as the only traded security is a one-period riskless one.

This economy is formally identical to the limited-participation economy of Basak and Cuoco (1998), except for a small difference in interpretation. In their interpretation, group 1 is endowed with the risky security called "equity" with dividend e. Group 1 has access to both securities, whereas group 2 has access to the riskless security only. In their setup, however, the risky security is effectively redundant since a group of identical agents (those of group 1) are the only ones having access to it. No trading of it actually takes place at any time. In Basak and Cuoco, the security is nonetheless "held", but only because agents of group 1 are endowed with it.[23] We can just as well consider this economy as an example of an incomplete-market economy.

Basak and Cuoco (1998) calculate analytically the equilibrium market prices of risk for the special case in which group 2 has logarithmic utility and receives no endowment. By our binomial method, we are able to generalize this economy to any pair of power utility functions. We consider the example in which the utility function of agent l ($l = 1, 2$) for time t is: $\rho^t \left(c_{l,t}\right)^{\gamma_l} / \gamma_l$. In the tradition of Cox, Ross and Rubinstein (1979), we call $\eta = u, d$ (for "up" and "down") the two successor nodes of a given node ξ of time t, with increments in e that mimick the geometric Brownian motion.

[23] The initial distribution of wealth determines whether an equilibrium exists: $W_{2,0}$ must be positive, but not so large that agents of group 1 could never repay their initial short position in the bond.

In this example, the equations system (15) particularizes to the following:

Flow budget constraint or "marketability" condition:

$$c_{1,t+1,u} + F_{1,t+1,u} = \theta_{1,t} + e_{1,t+1,u}; \qquad c_{1,t+1,d} + F_{1,t+1,d} = \theta_{1,t} + e_{1,t+1,d},$$

$$c_{2,t+1,u} + F_{2,t+1,u} = \theta_{2,t}; \qquad c_{2,t+1,d} + F_{2,t+1,d} = \theta_{2,t}$$

"Kernel"condition:

$$\frac{\frac{1}{2}\left(c_{1,t+1,u}\right)^{\gamma_1-1} + \frac{1}{2}\left(c_{1,t+1,d}\right)^{\gamma_1-1}}{\left(\omega \times e_t\right)^{\gamma_1-1}} = \frac{\frac{1}{2}\left(c_{2,t+1,u}\right)^{\gamma_2-1} + \frac{1}{2}\left(c_{2,t+1,d}\right)^{\gamma_2-1}}{\left((1-\omega) \times e_t\right)^{\gamma_2-1}}, \tag{16}$$

Market clearing:

$$\theta_{1,t} + \theta_{2,t} = 0,$$

where the future wealths $F_{1,t+1,u}$ and $F_{1,t+1,d}$, $F_{2,t+1,u}$ and $F_{2,t+1,d}$ are interpolated functions. The geometry of this equation system is illustrated in figure 3. It indicates strongly that the solution of the system exists and is unique. Once the solution for $\{\theta_{2,t}, c_{2,t+1,u}, c_{2,t+1,d}\}$ is found for a value of ω, the exiting financial wealths are calculated:

$$F_{l,t} = \frac{\rho}{\left(\omega_l \times e_t\right)^{\gamma_2-1}} \left\{ \frac{1}{2} \left[\left(c_{l,t+1,u}\right)^{\gamma_2-1} \times \left(c_{l,t+1,u} + F_{l,t+1,u}\right) \right] \right.$$
$$\left. + \frac{1}{2} \left[\left(c_{l,t+1,d}\right)^{\gamma_2-1} \times \left(c_{l,t+1,d} + F_{l,t+1,d}\right) \right] \right\}; \; l = 1, 2; \omega_1 + \omega_2 = 0. \tag{17}$$

The values of $F_{1,t}$ and $F_{2,t}$ are interpolated over ω as a preparation for the next time-step.

Interpolations of the functions are implemented using the `Interpolation` command of *Mathematica*. The command generates `InterpolatingFunctions` in which divided differences are used to construct piecewise interpolating Lagrange polynomials of order 3. Given boundedness of the functions to be interpolated, these work well as long as there are two agents in the economy and the interpolation is unidimensional.[24]

`InterpolatingFunctions` provide approximate values that are valid over a `Domain`. In our codes, we take measures to extend the domain to the entire interval $[0, 1]$.[25] When endowment streams only take strictly positive values, so that each agent could live alone, it is in principle possible to obtain the solution at $\omega = 0, 1$ simply by considering the cases where they do live alone. In the current example, agents of group 2 receive zero endowment and are not able to live alone. In order to handle that case, we use limits. As $\omega_l \to 0$ or 1, we estimate the limit of the set of functions that are carried backward by fitting a third-degree polynomials to the last four points of a grid of values of ω_l and refining the end part of the grid until the estimate no longer changes. More specifically, we consider the solution of the equation system of *all* nodes of time $T-1$ (at which point the equations contain zero values for $F_{l,t+1,\eta}$ and $S_{n,t+1,\eta}$). We start with an evenly spaced grid for ω_l of one hundred points covering $[0, 1]$, obtaining the solution of the system for each of the hundred points.[26] Then we gradually add points by successively subdividing the last two segments of the grid near the edges and solving the system again over these points, until the estimates of the limits for all nodes for all functions to be iterated remain within bounds set by a `PrecisionGoal` and

[24] If there were more agents and, therefore, more endogenous state variables to interpolate over, one would use the methods of Lyasoff (2008).

[25] We are grateful to a referee for pointing out that, in cases in which agents start at time 0 with a zero financial position, if their utility functions are unbounded below when consumption approaches zero and their endowments have a lower bound, it may be possible to bound away from 0 and 1 the consumption shares that result from trading. When considering the entire future time path of consumption and trades, it is clear that, for both agents, the utility of trading must be larger than the utility of autarky as, otherwise, there would be no trade. That argument is also spelled out in Duffie et al. (1994), page 765. However, it seems to be valid for stationary equilibria only. In other cases, intertemporal restrictions on the bounds can be written but they do not provide the specific values of the bounds at any given time.

[26] In calculating the solution at each point, we use a "predictor" based on the four previously calculated points, to provide the root-finding routine with an excellent initial solution.

an `AccuracyGoal`. When these goals are set at 10^{-20}, the grid typically accumulates ten to twenty additional points above 99/100 and below 1/100.

The same grid is then used repeatedly at all points in time. Once the grid is set up, calculation time is then proportional to the number of nodes. Figure 4, the top panel of which is analogous to figure 2 in Basak and Cuoco (1998), shows the price of risk or Sharpe ratio on the equity market against the time-0 distribution of consumption,[27] group 1 having a risk aversion of 2 and group 2 (Non Stockholders) a risk aversion of 6 and other parameters corresponding to the calibration of Mehra and Prescott (1985) as cited by Basak and Cuoco (1998). With these risk aversions, the target empirical level of 0.4 is not easily attained.

The bottom panel of the figure shows the Negishi map, the relationship between time-0 wealth and the time-0 distribution of consumption.

For a calculation over seven points in time ($T = 6$; $t = 0, 1, 2, \ldots, 6$), setting up the grid and the time-$(T-1)$ calculation requires 67 seconds and for the remaining periods the calculation requires 64 seconds in total on the Intel Centrino dual processor of a Lenovo 3000V200 laptop computer.

B. The "reverse" Basak-Cuoco equilibrium

Our second example application is slightly more involved than the first one because it requires the simultaneous backward induction of both functions F and S. For purposes of

[27] S being the quoted price for equity, the market price of risk on the equity market is:

$$\frac{\left(\frac{1}{2}\frac{\phi_{2,t+1,u}}{\phi_{2,t}} + \frac{1}{2}\frac{\phi_{2,t+1,d}}{\phi_{2,t}}\right) \times \left(\frac{1}{2}\frac{\delta_{t+1,u}+S_{t+1,u}}{S_t} + \frac{1}{2}\frac{\delta_{t+1,d}+S_{t+1,d}}{S_t}\right) - 1}{\frac{1}{2}\left(\frac{\delta_{t+1,u}+S_{t+1,u}}{S_t} - \frac{\delta_{t+1,d}+S_{t+1,d}}{S_t}\right)} = -\left[\frac{\phi_{1,t+1,u}}{\phi_{1,t}} - \left(\frac{1}{2}\frac{\phi_{2,t+1,u}}{\phi_{2,t}} + \frac{1}{2}\frac{\phi_{2,t+1,d}}{\phi_{2,t}}\right)\right]$$
$$= \left[\frac{\phi_{1,t+1,d}}{\phi_{1,t}} - \left(\frac{1}{2}\frac{\phi_{2,t+1,u}}{\phi_{2,t}} + \frac{1}{2}\frac{\phi_{2,t+1,d}}{\phi_{2,t}}\right)\right] = \frac{1}{2}\left[\frac{\phi_{1,t+1,d}}{\phi_{1,t}} - \frac{\phi_{1,t+1,u}}{\phi_{1,t}}\right]$$

illustration, we reverse the example of the previous subsection and consider an incomplete market in which there is no riskless asset available for trade. Instead, the risky equity alone, which pays $\delta = e$, is available for trade. Equity has a price S. The equations system (15) for that case is:

Flow budget constraint or "marketability" condition:

$$c_{1,t+1,u} + F_{1,t+1,u} = \theta_{1,t} \times (e_{t+1,u} + S_{t+1,u}) + e_{1,t+1,u}$$

$$c_{1,t+1,d} + F_{1,t+1,d} = \theta_{1,t} \times (e_{t+1,d} + S_{t+1,d}) + e_{1,t+1,d}$$

$$c_{2,t+1,u} + F_{2,t+1,u} = \theta_{2,t} \times (e_{t+1,u} + S_{t+1,u})$$

$$c_{2,t+1,d} + F_{2,t+1,d} = \theta_{2,t} \times (e_{t+1,d} + S_{t+1,d})$$

"Kernel "condition:

$$\frac{\frac{1}{2}(c_{1,t+1,u})^{\gamma_1 - 1} \times (e_{t+1,u} + S_{t+1,u}) + \frac{1}{2}(c_{1,t+1,d})^{\gamma_1 - 1} \times (e_{t+1,d} + S_{t+1,d})}{(\omega \times e_t)^{\gamma_1 - 1}}$$
$$= \frac{\frac{1}{2}(c_{2,t+1,u})^{\gamma_2 - 1} \times (e_{t+1,u} + S_{t+1,u}) + \frac{1}{2}(c_{2,t+1,d})^{\gamma_2 - 1} \times (e_{t+1,d} + S_{t+1,d})}{((1-\omega) \times e_t)^{\gamma_2 - 1}}$$

Market-clearing condition:

$$\theta_{1,t} + \theta_{2,t} = 0,$$

where the future wealths $F_{l,t+1,u}$ and $F_{l,t+1,d}$ are obtained again from the interpolated recursive financial-wealth formula (14) and the future prices of equity $S_{t+1,u}$ and $S_{t+1,d}$ are obtained from the interpolated recursive price formula (13).

As an illustration of the solution, we display in figure 5 the Sharpe ratio (market price of risk in the equity market) as a function of group 2's (the constrained group) share of consumption. The result is a "negative-equity premium" configuration.

C. Wu's example of buy-and-hold investors

Next, we examine an example that involves two endogenous state variables. Tao Wu (2006) has constructed an equilibrium in an economy that is similar to that of our first Basak-Cuoco example with the difference that agents of group 2 are no longer prevented from accessing the equity market. They access it but in a mechanical way, making each period a contribution (to their pension fund) with which equity is bought and held till the last period where they consume the payoff. The additional endogenous state variable, θ, is the fraction of equity held by the people of group 2. Parameter values are as in the Basak-Cuoco examples. The periodic contribution made by group 2 is 12% of output. We show in figure 6 the resulting market price of risk in the equity market (where only group 1 trades freely and sets prices) against group 2's share of aggregate consumption, at a time when their fraction of equity shares held is equal to 20%.[28]

In this example, the financial wealth of group 2 and the market price of equity are interpolated in two dimensions over the two endogenous state variables. Even though equity is not traded freely between the two groups, the function giving its price as set by group 1 is needed at each stage of the backward induction to determine, for a given amount of contribution, how many new shares group 2 acquires. As before, we have taken great care to interpolate with precision the functions over the entire domain $[0, 1]$ of the endogenous variable ω, which we had so far. However, we have allowed extrapolations over the new endogenous state variable θ; extrapolation occurs when group 2 holds more than 100% of the equity market (with short selling by group 1). The graphs (not shown) of the functions against θ justify to some extent this treatment: they are practically linear in the neighborhood of $\theta = 1$ and beyond.

[28] Figure 6 is very similar to figure 14 in Wu's article, which is, however, drawn for $T = 50$. Wu works out an approximation to a system of two continuous-time partial differential equations. He does not spell out the boundary conditions he uses at the edges of the numerical domain used for his functions.

The example demonstrates that the procedure can handle more than one endogenous state variable, although the computing burden is, of course, greatly increased. The example does not invalidate the idea we have put forward before, that state prices (or consumption shares) are natural state variables in an incomplete financial market in which investors trade optimally. In the current example, an additional state variable is needed because one category of traders acts mechanically, as opposed to rationally.

IV. Other examples and comparison with two other methods

We now present three additional examples, one of which is calibrated to the U.S. economy. We use these three examples also to compare our own algorithm to two others. In the first subsection, we run two experiments that allow a comparison with the global method. In the second subsection, we implement our method on the realistic calibration of Heaton and Lucas (1996) and comment on the solution technique used by these authors.

There is a third comparison that would be warranted but which cannot be carried out at this point. In a popular implementation of the recursive approach that is used a lot by macroeconomists, each agent solves his/her optimization problem given the anticipated behavior of aggregate state variables. Once the optimization problem is solved, the decisions of individual agents are simulated and aggregated. The aggregate is then regressed linearly on the aggregate state variables and the cross-population moments of the wealth distribution. More precisely, it proceeds as follows:[29]

1. Let endogenous, agent-specific state variables include accumulated wealth and asset prices.

[29] This is the algorithm used by den Haan (1997), Krusell and Smith (1998) and, in two Finance papers, by Gomes and Michaelides (2007) and Storesletten, Telmer and Yaron (2007).

2. Postulate agents' beliefs or forecasts in the form of some autoregressive dynamics for the aggregate state variables. This is the "expectations step".

3. Solve the individual investor's consumption-asset accumulation problem by dynamic programming, using either value-function or policy-function iteration and either first-order conditions or exhaustive enumeration. The solution is in the form of a polynomial obtained by perturbation or by projection.

4. Simulate the resulting economy, equating demand with supply to determine the values of the endogenous aggregate variables

5. Run linear regressions on the simulated numbers to estimate the resulting dynamics of the aggregate state variables. Stop if the dynamics so calculated are close to those postulated under 2. Otherwise, go back to 3.

In contrast to this implementation which is approximate to an unknown degree, our algorithm is exact and is presumably more efficient since, as explained before, we do away with the expectations step by solving directly for the forecast. At the same time, however, the algorithm of the macroeconomists is capable of handling a kind of idiosyncratic risk that hits differently each of a continuum of investors (assumed, however, to be otherwise identical), and that we have not addressed yet.

A. Examples that permit a comparison with the global method

1. Example #6.2 in Cuoco and He (2001)

As has been mentioned in the introduction, it is possible to stack all the first-order conditions (15) of all the nodes into one large system, add the time-0 equations (12) and then substitute into this system the recursions (14, 13). This huge system can conceivably be solved simultaneously in one fell swoop. We call this approach the "global method", as opposed to the recursive method, for the solution of the forward-backward system.

In their paper of 2001, Cuoco and He write and solve a large system of that type.[30] In their numerical example #6.2 (page 289), they consider a two-period ($t = 0, 1, 2$; $T = 2$) economy with two securities: a long-term bond (maturing at time 2) and the equity claim. The node of time 0 has three spokes. At time 1, one node has two spokes and the other two have three spokes. The initial condition imposed is that the net financial wealth of both groups be equal to zero.

In figure 7, we plot the solution we obtain by the recursive method for the points of our grid that lie in a neighborhood of zero initial financial wealth. We can read from the diagram that, at zero wealth of group 2, the time-0 equilibrium price for the bond is 0.946, while a similar picture for the price of equity would produce the number 2.070 and for the level of consumption of group 2 the number 1.172, exactly as in the article (page 291).

Admittedly, the global method, when it converges to a solution, provides a solution for a single value of initial wealth much faster than does the recursive method. It should be pointed out, however, that the recursive method delivers a whole set of points as in the figures above. A proper horse race between the two methods is meaningful only in the case where a wide range of points is required. For instance, in this example, the global solution delivered one point in 0.89 seconds. For the full grid of 127 points, 113 seconds would be required while the recursive method delivers them in 65 seconds. Of course, these comparisons are only indicative, as times needed to get a solution are very dependent on starting values provided to the root-finding routine.

[30] The system in question is equation (33) on page 285 of Cuoco and He (2001).

For the case in which the tree is binomial, we emphasize very strongly that, even when the exogenous state variables are Markovian, the global approach does not permit the use of a recombining tree. This is because a recombining node would have a unique value of the exogenous state variables but would correspond to two different values of the endogenous state variables, depending on which node the process is coming from. There is path-dependence. For this reason, the recursive method works with great advantage compared to the global method. Because of the possibility of recombination, there always exists a large enough number of periods T such that the number of nodes $T+1$ under recombination, multiplied by the number of grid points is less than the number of nodes 2^T under no recombination, thus allowing the recursive method to compute faster than the global one.

2. A gauntlet thrown at the global method

We now develop an example with a trinomial tree that recombines: 3 nodes become 5, then 7 etc.. The purpose of it is to demonstrate that our recursive method can cope with a situation that the global method would be unable to handle. Everything is as in the Basak-Cuoco example, except that now the agents can trade a risky security and a bond. There are two agents, the first one being endowed with a stream of dividends, while the second one is not endowed with any assets (other than the initial wealth). The dividends (with which the first agent is endowed) are the only output in the economy. The two agents trade one risky security (one share of which entitles its owner to the entire stream of dividends) and also trade a bond that pays one unit of numeraire in the next period. Both securities are in zero net supply. The only uncertainty in the economy is in the output (i.e., in the stream of dividends from the risky security) which spans a trinomial tree. The growth rates on that tree are: 0.982609, 1.017, 1.0526. The conditional probabilities on the tree are:

0.384602, 0.15, 0.465398. The tree is created for 20 periods. The total number of nodes on this recombining tree is: 484. We use an ω grid containing 67 points. At each node there are portfolio choices for agent 2, and consumption for agent 2 in the 3 consecutive nodes. The number of unknowns at each point is 8. The computation is repeated for each node and each gridpoint, which is $484 \times 67 = 32,428$ times. The calculation time for periods 0 to 19 came to 166.672 seconds.

If the global method had been used, the path dependence due to wealth would have prevented us from using a recombining tree. The total number of nodes would have been: $3^{20} = 3,486,784,401$. There would have that number times 3 unknowns, which would have come to: 10,460,353,203. Even if the high-performance solver used exploited the sparseness of the system, it would be inconceivable for it to solve a system with 10 billion unknowns.[31]

Even though the recursive method computes many points on the ω-grid that may never be visited given the initial conditions and, in this sense, is wasteful, for as long as the exogenous tree can be made recombining,[32] it will always win out, simply because, in the global method, the path-dependence prevents recombination.

B. The Heaton and Lucas (1996) example and "auctioneer" algorithm

The aim of our final illustration is to demonstrate that our technique can handle life-size applications of incomplete-market theory. For that, we use the model put together by Heaton

[31] The same point can be made about our next example of subsection B, in which each node has eight sucessors and which we solve over six periods. Without recombination, that would be $8^6 = 262,144$ separate nodes for the global method. Since each node in that example involves 16 unknowns, that would be a global system of 4,194,304 unknowns.

[32] It is often said that any process that is Markovian generates a filtration that can be coded on a tree that is recombining. But we are not aware of a general method that would provide the tree for any Markov multidimensional process with time-varying expected increments and variance-covariance matrix. And, to our knowledge, the general pattern of recombination has not been established. For a univariate example of the coding of an AR(1) process, see Tuckman (2002), page 238.

and Lucas (1996). This is an equilibrium with two classes of agents, an incomplete market, trading costs and borrowing constraints. Borrowing constraints are introduced by them on *a priori* grounds, such as restrictions due to (un-modeled) moral hazard, so that, in their formulation, portfolios used as state variables do have well-defined bounds. Here, we consider *only the incomplete-market aspect.*

The only two assets available are the short-term (one-period) riskless security and the equity.

The model is calibrated to match the U.S. economy, including idiosyncratic labor shocks observed on panel data. The two groups of households receive dividends in accordance with their shareholding and differ only in the allocation of output to their respective labor income. Otherwise, the households have *identical, constant relative risk aversions* $1-\gamma$ and discount rates ρ.

For that reason, wealth and price functions satisfy a second homogeneity property with respect to total output, in addition to the homogeneity with respect to current state prices that we pointed out in section II. Total output e_t is then a scale variable, which can be factored out and need not be explicitly included as an exogenous state variable. The scaled variables follow a stationary process whereas the unscaled ones do not. The re-scaling leaves three exogenous state variables that describe the exogenous aspects of the economy at any given time: (i) the rate of growth of output g_t realized between $t-1$ and t (which, with previous notation, would have been $e_t/e_{t-1,\xi^-}$), (ii) the share of output paid out as dividend, *vs.* wage, (iii) the share of wage bill that is paid to group 1, vs. group 2. These are driven by an eight-state ($K = 8$) Markov chain, whose transition probabilities $\pi_{t,t+1,\eta}$ are calibrated to U.S. data.[33] Dividends are called $\delta_{t+1,\eta}$. Wages paid to group l are called $e_{l,t+1,\eta}$. We

[33] This is literally a Markov chain, not just any Markov process: the same eight states recur at all times so that, by definition, they recombine.

introduce one endogenous state variable ω defined as the share of group 1's current (time-t) consumption in current output, as in the previous examples.

Redefining time-$(t+1)$ variables $c_{1,t+1,\eta}$, $e_{1,t+1,\eta}$, $c_{2,t+1,\eta}$, $e_{2,t+1,\eta}$, $\delta_{t+1,\eta}$ to have the meaning they had in previous sections except that they refer to amounts *per unit of time-t output*, the system can be written as follows:

Flow budget constraint or "marketability" condition:

$$c_{1,t+1,\eta} - e_{1,t+1,\eta} + g_{t+1,\eta} \times F_{1,t+1,\eta} = \theta_{1,t,1} + \theta_{1,t,2} \times \left(\delta_{t+1,\eta} + g_{t+1,\eta} \times S_{t+1,\eta}\right), \quad 1 \leq \eta \leq 8\,,$$

$$c_{2,t+1,\eta} - e_{2,t+1,\eta} + g_{t+1,\eta} \times F_{2,t+1,\eta} = \theta_{2,t,1} + \theta_{2,t,2} \times \left(\delta_{t+1,\eta} + g_{t+1,\eta} \times S_{t+1,\eta}\right), \quad 1 \leq \eta \leq 8\,,$$

"Kernel "condition for short-lived riskless asset:

$$\frac{1}{\left(\omega_t\right)^{\gamma-1}} \sum_{\eta=1}^{8} \pi_{t,t+1,\eta} \times \left(c_{1,t+1,\eta}\right)^{\gamma-1} = \frac{1}{\left(1-\omega_t\right)^{\gamma-1}} \sum_{\eta=1}^{8} \pi_{t,t+1,\eta} \times \left(c_{2,t+1,\eta}\right)^{\gamma-1}$$

"Kernel "condition for equity:

$$\frac{1}{\left(\omega_t\right)^{\gamma-1}} \sum_{\eta=1}^{8} \pi_{t,t+1,\eta} \times \left(c_{1,t+1,\eta}\right)^{\gamma-1} \times \left(\delta_{t+1,\eta} + g_{t+1,\eta} \times S_{t+1,\eta}\right)$$

$$= \frac{1}{\left(1-\omega_t\right)^{\gamma-1}} \sum_{\eta=1}^{8} \pi_{t,t+1,\eta} \times \left(c_{2,t+1,\eta}\right)^{\gamma-1} \times \left(\delta_{t+1,\eta} + g_{t+1,\eta} \times S_{t+1,\eta}\right)$$

Market-clearing condition:

$$\theta_{1,t,1} + \theta_{2,t,1} = 0; \theta_{1,t,2} + \theta_{2,t,2} = 1.$$

As usual, the undiscounted financial wealth of group 2 and the equity price are defined recursively:

$$
\begin{aligned}
F_{l,t} &= \frac{\rho}{\omega_l^{\gamma-1}} \sum_{\eta=1}^{8} \pi_{t,t+1,\eta} \times (c_{l,t+1,\eta})^{\gamma-1} \times (c_{l,t+1,\eta} - e_{l,t+1,\eta} + g_{t+1,\eta} \times F_{l,t+1,\eta}) \\
F_{l,T} &\equiv 0; l = 1, 2 \\
S_t &= \frac{\rho}{(1-\omega_t)^{\gamma-1}} \sum_{\eta=1}^{8} \pi_{t,t+1,\eta} \times (c_{2,t+1,\eta})^{\gamma-1} \times (\delta_{t+1,\eta} + g_{t+1,\eta} \times S_{t+1,\eta}) \\
S_T &\equiv 0
\end{aligned}
$$

We solve the problem over seven points in time ($T = 6; t = 0, ..., 6$). The time required is 678 second to set up the grid and calculate for one period and 400 seconds for each additional period of time.

We show in figure 8 the equilibrium Sharpe ratios on the equity security as functions of the share $(1 - \omega)$ of consumption of agent 2 in Heaton and Lucas's four "low-realized growth" states and in figure 9 the same in the four "high-realized growth" states. Neither the "equity-premium" nor the "excess-volatiliy" puzzle is solved by this specification. That is the reason for which Heaton and Lucas say that debt constraints and frictions are needed to match the moments observed in the data.

Heaton and Lucas calculate the stationarized equilibrium numerically, by means of a tatonnement – or "auctioneer"– algorithm described in Lucas (1994) and based on the primal program (2) of each investor and the condition that supply equals demand in the financial market. The state variables are the portfolios of households. The auctioneer's algorithm works as follows:

1. The portfolios are restricted to a discrete 30×30 grid.

2. One copy of the grid is attached to each of the 8 states of the exogenous Markov chain and an extended Markov chain with a total of $30 \times 30 \times 8$ states is defined. A "state" in the extended Markov chain is a combination of values of the three exogenous state variables plus a portfolio point on the grid.

3. A portfolio policy is a map of the grid into itself. This means a choice of an exiting portfolio for a given entering portfolio. All portfolio policies under consideration are assumed time-independent.

4. Any particular portfolio policy gives rise to a large transition probability matrix for the extended Markov chain.

5. The price of each security is defined as a function on the 30×30 grid. These pricing maps are time-independent.

6. The consumptions are solved for from the flow budget constraints, i.e., every concrete choice of portfolio and pricing maps uniquely defines consumptions from the flow budget constraint. There is now consumption attached to every state of the extended Markov chain (provided that the pricing maps are given).

7. An initial guess for the portfolio policy is made and the ratios of the marginal utilities (the state prices) are calculated from 6.

8. For any given portfolio policy, the pricing map for each agent is calculated implicitly from equating the security prices with the values of the next period payoffs, priced according to that agent's state prices (because of the time invariance of the prices, the pricing map appears on both sides of this relation, so that the pricing rule is actually an equation).

9. If the agents price the securities differently, an adjustment is made in the portfolio policy and step 8 is repeated followed by 9.

The assumptions of time independence of the portfolio policies and price maps are partly justified by the assumption that investors have an infinite time horizon but they also require that entering portfolios be sufficient endogenous variables to render Markovian the extended (partially endogenous) process and that a stationary equilibrium of the extended system exist. In her 1994 paper D. Lucas says:

"The assumed initial distribution of stock and bond holdings exhibits substantial persistence over time. For instance, dependence on the initial distribution is apparent even after 250 years. However, examination of the distribution of asset holdings after 1000 years and 5000 years, starting from various initial distributions of wealth, suggests that there exists a unique stationary distribution of portfolio holdings ...."

Our algorithm obviously does not require stationarity.

Recall also that, without *a priori* constraints, the range of possible portfolio values may not be known and may not even be in one piece (i.e., some portfolio combinations might be impossible), in which case the auctioneer algorithm would not be applicable whereas ours would be. At a more practical level, we can also point out that it would be difficult to extend the auctioneer method beyond two assets. By contrast, the degree of complexity of our calculation depends mostly on the number L of agents and not on the number of securities N. Adding more securities to the economy only increases the number of "kernel-condition" equations in the system that must be solved at each node.[34]

[34] Since we provide our root-finding routine with excellent predictors of the solution (see Footnote 26), the computing time remains very short even with an increased number of unknowns.

V. Multiplicity of solutions

There is no reason to believe that our nodal system (15) has a unique solution. In fact, when solving the Heaton-and-Lucas example in the previous section, we encountered a multiplicity when we went beyond seven points in time ($T = 6$). Adding one more period ($T = 7$), produced at $t = 0$, a situation described by figures 10 and 11.

Figure 10 displays group 2's share of future consumption in the eight successor states in relation to group 2's share of consumption when the time-0 state is state 1. Observe first that the relationships are not monotonic: when group 2's share today in state 1 is reduced to about 0.025, its share of consumption in future state 8 suddenly rises all the way to 0.8 while its amount of consumption in future state 4 suddenly drops more moderately. Referring back to the Markov-chain data in the article by Heaton and Lucas (1996), table 2, page 455, one notices that the probability of a transition from state 1 to state 8 is very low and equal to 0.002.[35] Group 2 can afford a huge share of consumption in state 8 in exchange for a moderate drop in state 4 only because state 8 has extremely low probability of occurring. The observation illustrate a basic principle of financial markets: very large transfers of consumption from one group to another occur endogenously in low-probability events.

Figure 11 displays the same graphs as figure 10 but on a much larger scale for the interval $[0, 0.05]$ of group 2's consumption share today. It is apparent that the relationships for most of the future states display a jump combined with a "beak" or a "cusp".[36] Over a small

[35] The Heaton-and-Lucas Markov chain is symmetric: swapping groups 1 and 2 and swapping simultaneously states 1-4 with states 5-8 gives back the same process. While we focus our observations on the transitions from state 1 – and especially that from state 1 to state 8,– there are symmetric observations to be made about transitions from state 5 and especially that from state 5 to state 4.

[36] We made every effort to show that the "cusp" was only a numerical aberration due either to an insufficiently fine grid or insufficient interpolation precision at previous stages of the calculation. But, by using a working precision of 10^{-100} and after refining the grid and the interpolation, starting each time again from

interval of today's consumption values, for a given value of that consumption, there exist two possible solutions or temporal equilibria for the consumptions of tomorrow in the various states, which means that future consumption is not a function of today's consumption; it is generally a correspondence. The equilibrium "hesitates" between letting group 2 consume in state 8 or in state 4. The source of the phenomenon is the non monotonicity of the eight stock-price functions (not shown), which are U-shaped in each of the eight future states. As one lengthens the horizon, the stock being a security that lives to the horizon behaves more and more differently from the short-term rate of interest. The stock-price U-shaped functions become flatter and take a sharper turn towards low and high shares of consumption.

One could not continue the backward induction beyond this many periods unless one had a way of making a selection among the several nodal equilibria as being the one that investors would earlier expect to prevail at that time. The selection cannot be made on the basis of the levels of utility reached by the two groups since one should not expect incomplete-market equilibria with long-lived securities to have any property of optimality anyway.[37] Instead of a selection, Blume (1979) has proposed a randomization across equilibria. The idea of randomization has been applied to prove existence of stationary equilibria by Duffie *et al.* (1994) and to compute them by Feng *et al.* (2009). Some work remains to be done to sort out this issue in our setting. We emphasize that all methods that are exact would fail to easily provide the full answer in that situation anyway.

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the terminal point in time, till there was no longer any difference across the entire $[0, 1]$ interval in all the functions involved to a precision of 10^{-17}, there was no more room for doubt. The cusp is there.

[37] See Magill and Quinzii (1996), page 271.

VI. Conclusion

The equilibrium calculation method developed here opens at least three potential avenues of research.

The first and most immediate application will be to use the algorithms we have developed to follow the lead of Gomes and Michaelides (2008) in answering the question we raised in the introduction. It is important to find out whether incomplete-market equilibria can deliver a match between model and financial-market data. Missing-market risks should increase risk premia and volatility and cause the distribution of wealth in the investor population to act as a dimension of risk, separate from aggregate wealth. We are now equipped to determine what are likely orders of magnitude of these effects.

The second order of business will be to deal with equilibrium in the presence of transactions costs. In such an equilibrium, there will be periods of time during which, and states of nature in which no trade will take place and thus no price will prevail. It would be, therefore, impossible to embark on a direct calculation of equilibrium by tatonnement since the form of the process for prices, being of the intermittent kind (it is a "point process"), would be hard to specify *ab initio*. Cvitanic and Karatzas (1996), Cvitanic (1997) and Kallsen and Muhle-Karbe (2007) have shown how the dual approach can be applied to portfolio optimization under transactions costs. It can be extended to equilibrium, because, when information arrives at each node, the dual variables, unlike actual prices on trades, can be postulated to take values at all times and all nodes.[38] As we apply the binomial tree technique and as we progressively subdivide the time interval between nodes, it will be fascinating to see the manner in which the intermittent process for asset prices approaches a continuous process.

[38] Jouini and Kallal (2001) have already established some properties of the dual variable process.

Default risk is the third application to be considered. In a complete market, all risks being hedgeable, default can occur only when an economic agent chooses not to pay what he owes and to suffer the consequences.[39] In such a setting, agents default in states of nature in which they have a lot of debt but have received a large cash flow (the "take-the-money and run" kind of default). It is clear that reality does not fit that model: people are sometimes in situations where they "cannot" pay, because they must maintain a survival level of consumption. These can occur only in incomplete markets.

In the approach we have proposed, one has to recognize a number of endogenous state variable equal to the number of agents in the economy (minus one). The extension to produce an approximation valid for large populations is a very serious challenge. Krusell and Smith (1998) have provided such an approximation for the case of independent idiosyncratic risk across a totally homogeneous population: the mean of the distribution of wealth is then a sufficient state variable. In more general cases, the matter will be more complex.

Within a decade, mankind will want to devote as much computing power to the large-scale modelling of financial markets as is devoted today to the analysis of the earth's weather and atmosphere. We hope that our method will facilitate that undertaking.

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[39] In Alvarez and Jermann (2000), agents are kept under check so that they do not default, should they wish to, but the idea is similar.

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Appendix: Proof of Theorem 1

Proof. Given that the principle of dynamic programming applies, if the consumption plan c_l and the trading strategy θ_l attain agent-l's objective, then one can claim that $x^* = c_{l,t}$ and $y^* = \theta_{l,t}$ solve the "primal" optimization problem:

$$\begin{aligned} &\max_{x,y}\, G_{l,t}(x,y) \triangleq U_{l,t}(x) + \mathbb{E}_t\left[V_{l,t+1}\left[y\cdot(S_{t+1}+\delta_{t+1})\right]\right] \\ &\quad\text{subject to:}\;\; x + y\cdot S_t = e_{l,t} + W_{l,t}, \quad x\in\mathbb{R}_{++},\; y\in\mathbb{R}^N. \end{aligned} \tag{18}$$

Since $x > 0$, i.e. consumption is strictly positive, the Lagrangian for this problem is given by

$$\begin{aligned} \mathcal{L}_{l,t}(x,y,\lambda) = G_{l,t}(x,y) + \lambda\times\left(e_{l,t} + W_{l,t} - x - y\cdot S_t\right), \\ x\in\mathbb{R}_{++},\; y\in\mathbb{R}^N,\; \lambda\in\mathbb{R}\,. \end{aligned}$$

Since the left side of the only constraint in (18) is a linear function of $(x,y)\in\mathbb{R}_{++}\times\mathbb{R}^N$ with gradient (treated as a vector column) $\nabla(x+y\cdot S_t) = \{1, S_t\}\in\mathbb{R}^{1+N}$, the first order conditions (3) imply the following relation

$$\nabla G_{l,t}(c_{l,t},\theta_{l,t}) = \phi_{l,t}\times\{1,S_t\}. \tag{19}$$

Consider next the quantities $c_{l,t}$, $\theta_{l,t}$ and $\phi_{l,t}$ as functions of the entering wealth $W_{l,t}$, which are defined implicitly from (3) (we assume that $\theta_{l,T} = \mathbf{0}$). After differentiating both sides in (19) we get

$$\nabla^2 G_{l,t}\left[c_{l,t}(W_{l,t}),\theta_{l,t}(W_{l,t})\right]\cdot\{c'_{l,t}(W_{l,t}),\theta'_{l,t}(W_{l,t})\} = \phi'_{l,t}(W_{l,t})\times\{1,S_t\},$$

and this implies that

$$
\begin{aligned}
&\left\{c'_{l,t}(W_{l,t}), \theta'_{l,t}(W_{l,t})\right\}^{\intercal} \cdot \nabla^2 G_{l,t}\left[c_{l,t}(W_{l,t}), \theta_{l,t}(W_{l,t})\right] \cdot \left\{c'_{l,t}(W_{l,t}), \theta'_{l,t}(W_{l,t})\right\} \\
&\quad = \phi'_{l,t}(W_{l,t}) \times \left(c'_{l,t}(W_{l,t}) + S_t \cdot \theta'_{l,t}(W_{l,t})\right) = \phi'_{l,t}(W_{l,t}),
\end{aligned} \tag{20}
$$

where we have used the identity

$$
c'_{l,t}(W_{l,t}) + \theta'_{l,t}(W_{l,t}) \cdot S_t = 1\,, \tag{21}
$$

which is obtained by differentiating both sides of the constraint

$$
c_{l,t}(W_{l,t}) + \theta_{l,t}(W_{l,t}) \cdot S_t = e_{l,t} + W_{l,t}.
$$

For some fixed $1 \leq l \leq L$, consider the entire system of first order conditions (5) at all nodes $\xi \in \mathbb{F}_t$, $0 \leq t \leq T$. It is clear from the terminal condition: $V_{l,T}\left(W_{l,T}\right) = U_{l,T}\left(c_{l,T}\right) \equiv U_{l,T}\left(e_{l,T} + W_{l,T}\right)$ that the value function $V_{l,T}(\cdot)$ is strictly concave in any state $\sigma \in \Omega$, i.e., all value functions $V_{l,T,\sigma}(\cdot)$, $\sigma \in \Omega$, are strictly concave. Now suppose that for some $0 \leq t < T$ one can claim that the value functions $V_{l,t+1,\eta}(\cdot)$ are strictly concave, for all possible choices of $\eta \in \xi^+$ and $\xi \in \mathbb{F}_t$. Then the function

$$
\mathbb{R}_{++} \times \mathbb{R}^N \ni (x, y) \longrightarrow G_{l,t}(x, y) \in \mathbb{R},
$$

which was defined in (18), also must be strictly concave in state $\xi \in \mathbb{F}_t$. Since the security prices are non-negative, (21) implies that the vector

$$
\left\{c'_{l,t}(W_t), \theta'_{l,t}(W_t)\right\} \in \mathbb{R}^{1+N}\,,
$$

cannot vanish. In conjunction with the strict concavity of $G_{l,t}(\cdot,\cdot)$, (20) and (4) imply that in state $\xi \in \mathbb{F}_t$ one must have

$$V''_{l,t}(W_t) = \phi'_{l,t}(W_t) < 0.$$

The fact that the value functions $V_{l,t,\xi}(\cdot)$, $\xi \in \mathbb{F}_t$, are strictly concave for any $0 \leq t \leq T$ now follows by induction. As a result, we can claim that all functions $G_{l,t,\xi}(\cdot,\cdot)$ are strictly concave and that, therefore, the first-order conditions in (3) are both necessary and sufficient and, furthermore, cannot be satisfied with more than one choice for (c_l, θ_l, ϕ_l), $1 \leq l \leq L$. Finally, taking into account (4), these first-order conditions can be written in the form (5). ■

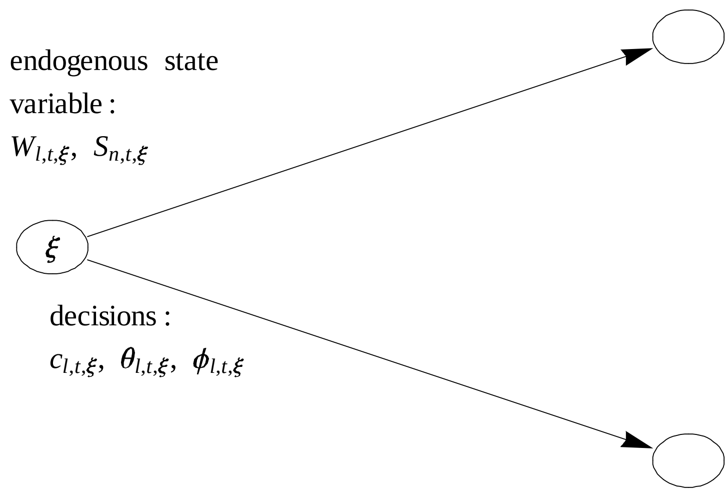

Figure 1. In the primal dynamic-programming formulation, when investor l is faced at time t (in state $\xi \in \mathbb{F}_t$) with entering wealth $W_{l,t,\xi}$, local price vector $S_{t,\xi}$, and new endowment $e_{l,t,\xi}$, he computes his immediate consumption $c_{l,t,\xi}$, his immediate trading strategy $\theta_{l,t,\xi}$ and his local (in time and state of the economy) Arrow-Debreu shadow prices $\phi_{l,t,\xi}$.

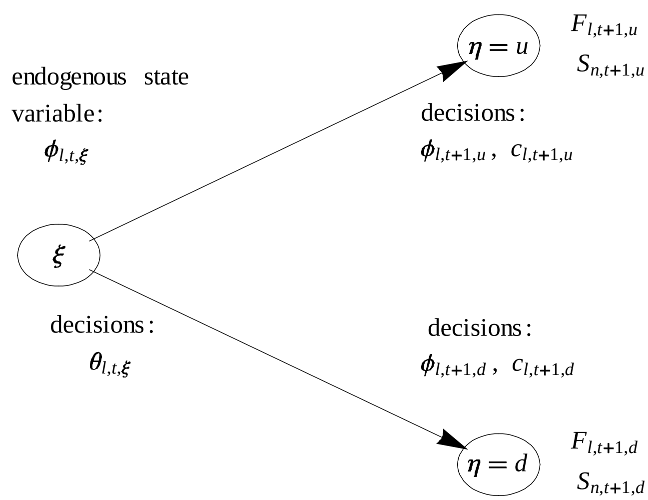

Figure 2. In the dual formulation, after a time shift of one equation, we now associate with the time-t node, the choice of consumption in the successor nodes of time $t+1$, given state prices at time t. This picture should be contrasted with figure 1.

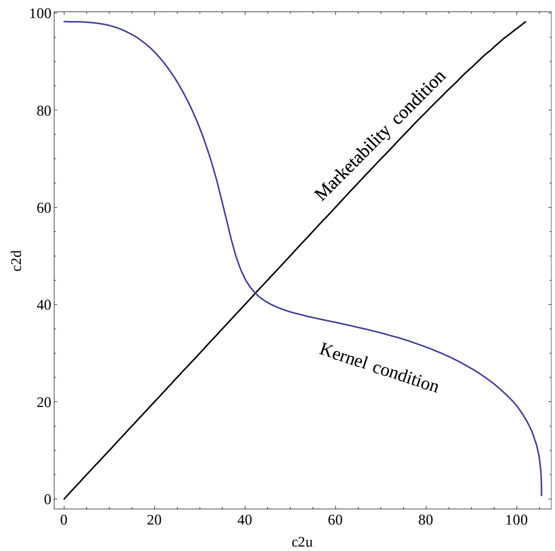

Figure 3. The geometry of the equations system (16) is displayed here. After substituting out $c_{1,t+1,u}$ and $c_{1,t+1,d}$ from the aggregate-resource constraint: $c_{1,t+1,u} = -c_{2,t+1,u} + e_{t+1,u}; c_{1,t+1,d} = -c_{2,t+1,d} + e_{t+1,d}$, the system has two remaining equations: the marketability condition and the kernel condition. The picture displays the loci of points at which each of these equations holds. The picture is calculated for parameter values: $T = 6, \gamma_2 = -5, \gamma_1 = -1, \beta = 0.999, \sigma_\delta = .0357, \mu_\delta = 0.0183$ and for the particular point $\omega = \frac{29}{50}$. The kernel condition alone is affected by the particular choice of ω (the locus shifts down as ω increases). The axes of the picture cover the entire ranges $c_{2,t+1,u} \in [0, e_{t+1,u}]$, $c_{2,t+1,d} \in [0, e_{t+1,d}]$.

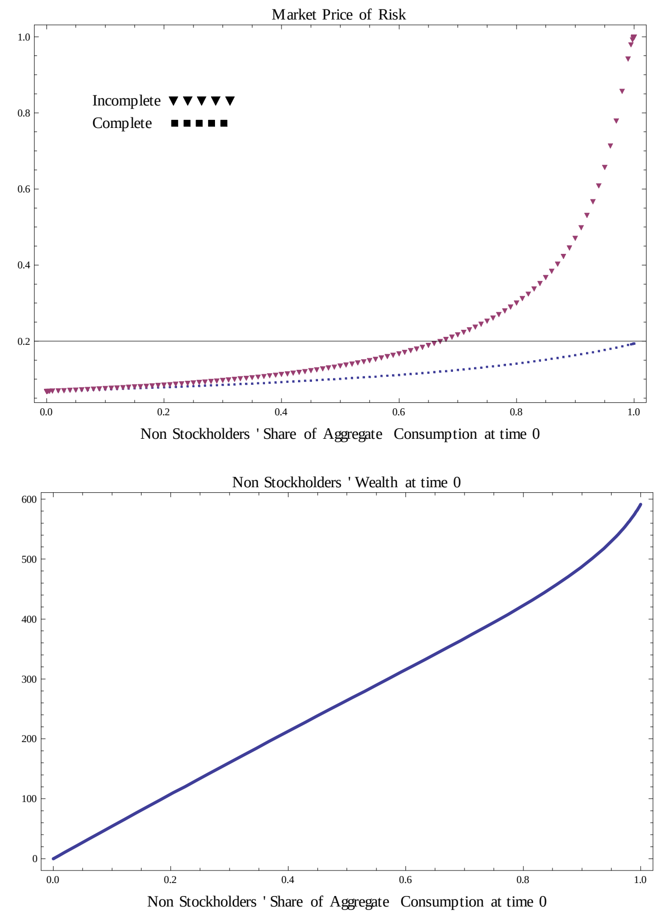

Figure 4. Basak-Cuoco equilibrium. The top panel of this figure shows the market price of risk applicable in the equity market where Group 1 alone "trades". Parameter values are: $T = 6, \gamma_2 = -5, \gamma_1 = -1, \beta = 0.999, \sigma_\delta = .0357, \mu_\delta = 0.0183$. The lower of the two curves, which corresponds to the complete-market case, is provided for comparison. The bottom panel of the figure is a **Negishi map**: it shows the relationship between time-0 wealth and the time-0 distribution of consumption, which is endogenous to wealth.

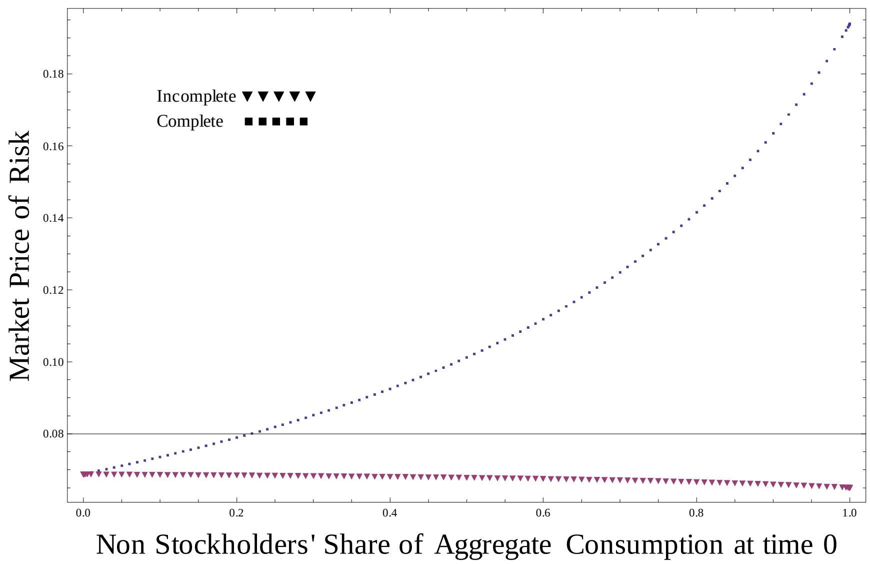

Figure 5. Reverse Basak-Cuoco equilibrium. This figure shows the market price of risk applicable in the equity market where both groups trade. Parameter values are: $T = 6, \gamma_2 = -5, \gamma_1 = -1, \beta = 0.999, \sigma_\delta = .0357, \mu_\delta = 0.0183$. The higher of the two curves, which corresponds to the complete-market case, is provided for comparison.

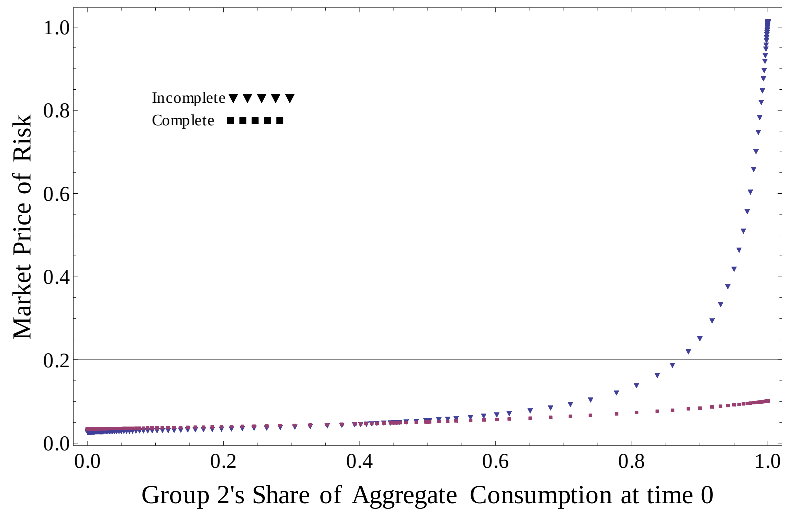

Figure 6. Wu's buy-and-hold investors: This figure shows the market price of risk applicable in the equity market where group 1 only trades freely, while group 2 invests there mechanically in a buy-and-hold fashion. On the x-axis is group 2's share of aggregate consumption. Parameter values are: $T = 3, \gamma_2 = -5, \gamma_1 = -1, \beta = 0.999, \sigma_\delta = .0357, \mu_\delta = 0.0183$. The period contribution made by group 2 is equal to 12% of output. The top line corresponds to a fraction of equity held by group 2 that is equal to 20% and the bottom line to the complete-market situation (with optimal holdings).

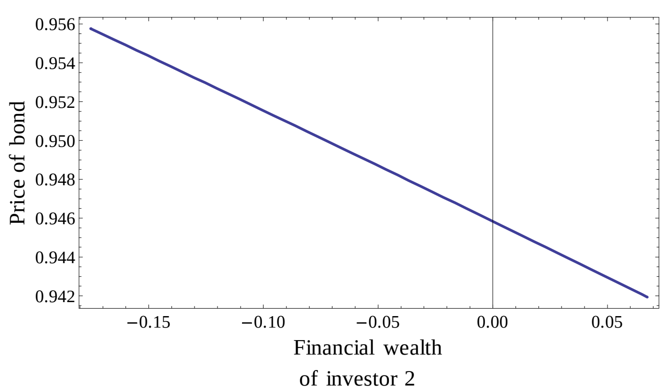

Figure 7. Cuoco-He (2001) Example #6.2: The intersection of the line of points with the y-axis gives the price of the bond corresponding to the solution of Cuoco and He (2001), Page 291.

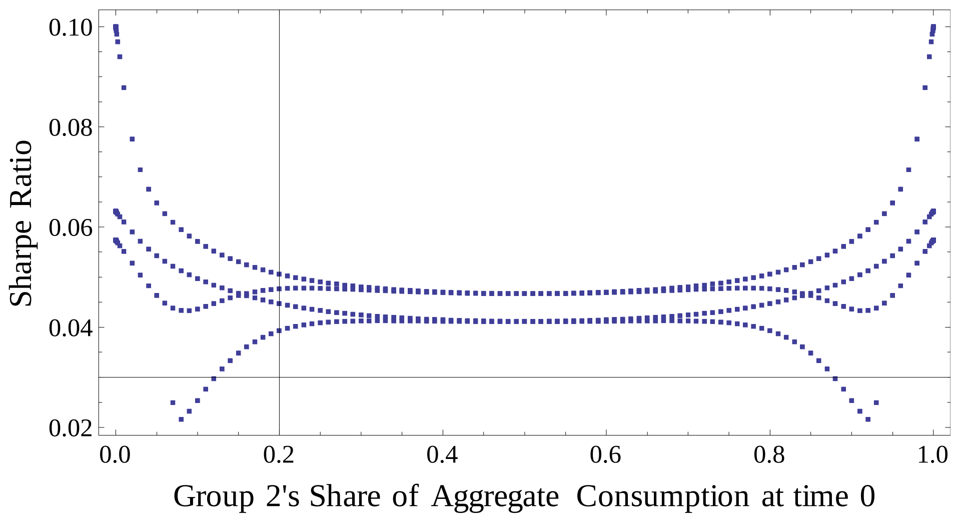

Figure 8. Low-growth states: This figure shows the Sharpe ratio on the equity security when the two groups of agents only trade the Bill and the equity, depending on the state of nature the economy is in. This figure contains the four states of nature in which the realized growth rate is low. On the x-axis is the fraction of output consumed by group 2. Parameter values are as in Heaton and Lucas (1996), table 2, page 455.

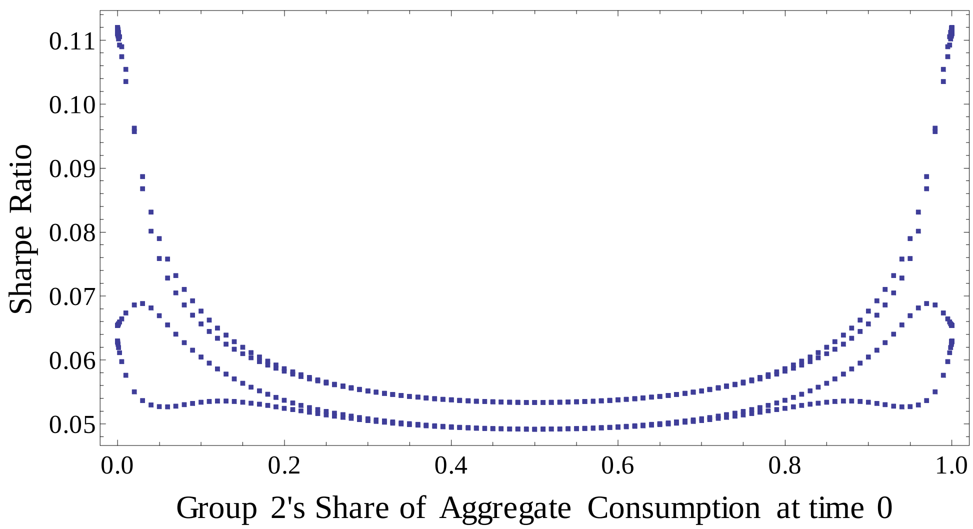

Figure 9. High-growth states: This figure shows the Sharpe ratio on the equity security when the two groups of agents only trade the Bill and the equity, depending on the state of nature the economy is in. This figure contains the four states of nature in which the realized growth rate is high. On the x-axis is the fraction of output consumed by group 2. Parameter values are as in Heaton and Lucas (1996), table 2, page 455.

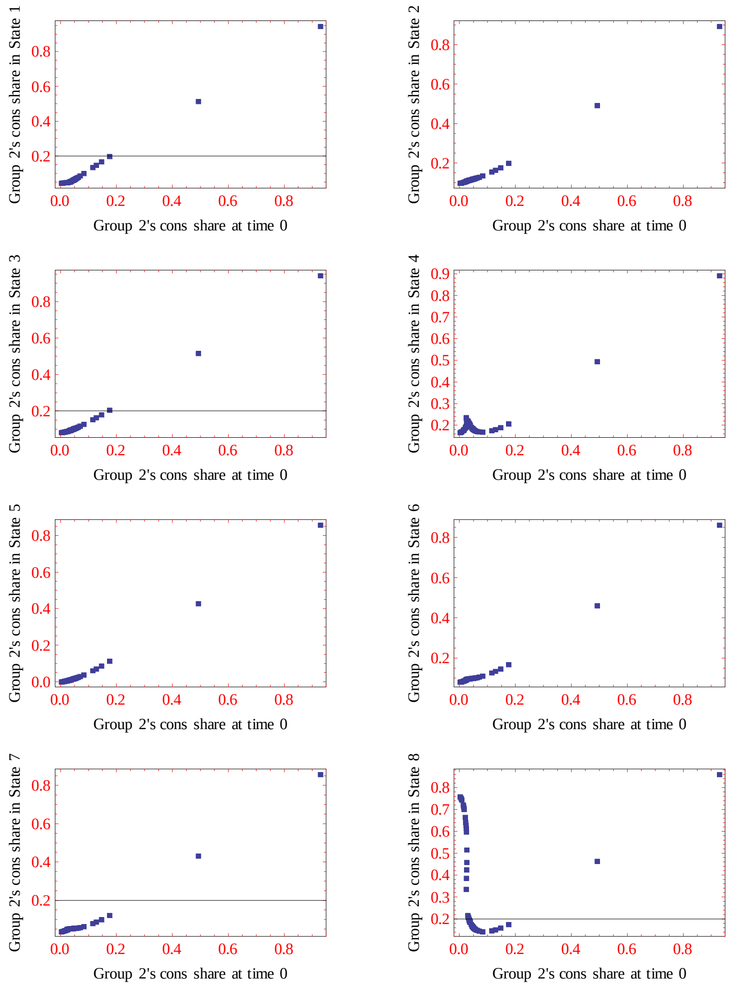

Figure 10. Consumption correspondence: This figure shows on the y-axis the consumption share of group 2 in each of the eight states at time 1. On the x-axis is the fraction of output consumed by group 2 at time 0. Parameter values are as in Heaton and Lucas (1996), Table 2, page 455 and $T = 7$, $\gamma = 0.5$. Note: here the x-grid has been chosen to focus the diagram on the singular point towards the left of the x-axis. The grid normally used for computations is more evenly spaced.

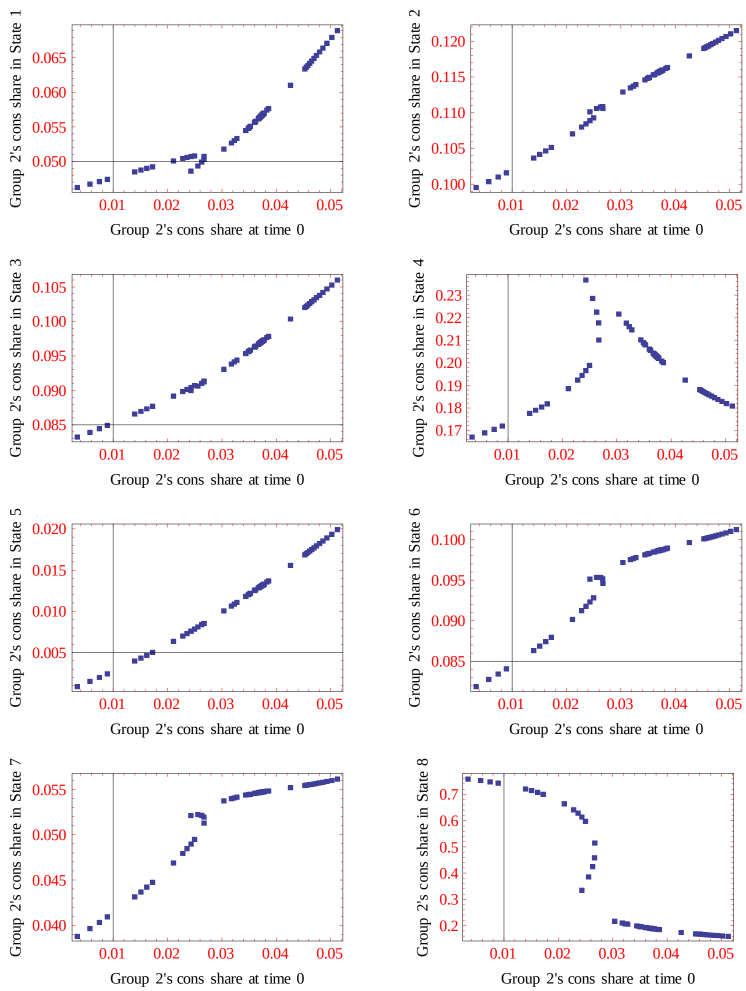

Figure 11. Consumption correspondence (large scale): This figure shows on the y-axis the consumption share of group 2 in each of the eight states at time 1. On the x-axis is the fraction of output consumed by group 2 at time 0. Parameter values are as in Heaton and Lucas (1996), Table 2, page 455 and $T = 7$, $\gamma = 0.5$.