# FINANCIALLY CONSTRAINED ARBITRAGE

# & CROSS-MARKET CONTAGION

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## Limits of Arbitrage

- **Challenges to asset pricing theory**
  - Market anomalies:
    - Deviations from the Law of One Price.
    - Predictability of asset returns.
  - Contagion and liquidity linkages.
  - Financial crisis.

- **Leading approaches**
  1. Refinements of standard theory (preferences,...)
  2. Behavioral Finance.
  3. Frictions/Transaction costs:
     (a) **Financial constraints.**
     (b) Agency problems.
     (c) Search frictions.

## Financially Constrained Arbitrage

- **Main ingredients**

  1. *Arbitrageurs are "special":*
     – Specialised institutions (I-banks, primary dealers,...).
     – No close/fast substitutes.
  2. *Arbitrageurs face financial constraints.*

$$\text{Arbitrage Capital} \Rightarrow \text{Asset Prices and Liquidity} \Rightarrow \text{Arbitrage Capital}$$

- **Implications**

  – Investment policy.
  – Asset prices and market liquidity.
  – Welfare and Policy.

## Examples

- **Stocks + Market Makers**
  - MM: Higher inventory, low revenues.
  - $\Rightarrow$ Lower daily stock market liquidity + Contagion across different stocks.

- **Currencies + Hedge Funds**
  - Carry Trade: Borrow/invest in low/high interest rate country.
  - Lower AUM + Greater outflows.
  - $\Rightarrow$ Interest rate gap widens + Low interest rate currency appreciates.
  - Fall 2008:
    - Large outflow from hedge funds.
    - $\Rightarrow$ Low interest rate currencies appreciated (e.g. Yen vs. GBP).

## Main Contribution

### Framework

- Dynamics + Multiple assets.
- Nests standard asset pricing model.

### Riskfree arbitrage

- Closed form ⇒ Many properties.

### Risky arbitrage

- Amplification.
- Contagion.
- Stabilizing vs. destabilizing effect.
- Non-monotonic effect of arbitrage capital on liquidity, volatility, correlations.

## Literature

- Pre-1998 crisis: Tuckman and Vila (1992), Shleifer and Vishny (1997)

- Post-1998 crisis: Basak and Croitoru (2000, 2006), Xiong (2001), Liu and Longstaff (2004), Pavlova and Rigobon (2008), Zigrand (2006), Rahi and Zigrand (2007), Krishnamurthy and He (2009a, 2009b), Kondor (2009), Duffie and Strulovici (2009).

- More to come.

- Survey in Gromb and Vayanos (2010).

- Closest papers:
  - Kyle and Xiong (2001): No constraints.
  - Gromb and Vayanos (2002): Single arbitrage opportunity.
  - Brunnermeier and Pedersen (2007): Static.

# Roadmap

- **Model.**
- Riskfree arbitrage.
- Risky arbitrage.

# MODEL

- Continuous time, infinite horizon ($t \in \mathbb{R}^+$).

## Assets

- Riskless asset with exogenous return $r$.

- Pairs of risky assets $(i, -i) \in \mathcal{I}^2$
  - Zero net supply.
  - Dividends

$$
\begin{aligned}
dD_{i,t} &= D_i dt + \sigma_i dB_{i,t} + \sigma_i^f dB_{i,t}^f \\
dD_{-i,t} &= D_i dt + \sigma_i dB_{i,t} - \sigma_i^f dB_{i,t}^f
\end{aligned}
$$

## Regular Investors

- **Market segmentation**
  - Some investors can only hold the riskfree asset and risky asset $i$.
  - Competitive, measure $1$, initial wealth $w_{i,t}$.
  - Short-lived OLG: $(i,t)$-investors live over $[t, t+dt]$:

$$\max \quad \mathsf{E}_t(dw_{i,t}) \quad - \quad \frac{a_i}{2} \cdot \mathsf{Var}_t(dw_{i,t})$$

- **Liquidity demand**
  - At time $t$, $(i,t)$-investors receive (the equivalent of) $u_{i,t}$ shares of asset $i$.
  - Opposite shocks for $(i,t)$- and $(-i,t)$-investors:

$$u_{-i,t} = -u_{i,t}$$

  - $\Rightarrow$ Unrealized risk-sharing gains from trade.

## Arbitrageurs

- **"Special" liquidity providers**
  - Infinitely-lived, competitive, measure $1$

$$\max \quad \mathsf{E}_t \left[ \int_t^{\infty} \log(c_s)\, e^{-\beta(s-t)}\, ds \right] \qquad \text{with } \beta > 0$$

  - Can invest in all assets ⇒ Buy cheap asset/Sell pricy asset ⇒ Provide liquidity.

- **Financial constraint (reduced form)**
  - Long or short $1$ share of asset $i$ or $-i$ ⇒ Haircut in cash $m_i > 0$.
  - ⇒ Arbitrageurs' wealth $W_t$ and positions $x_{i,t}$ must satisfy

$$\boxed{\sum_{i \in \mathcal{I}} m_i |x_{i,t}| \leq W_t}$$

## Symmetric Equilibrium

**Definition:** *Symmetric equilibrium:*

- *Arbitrageurs (optimally) enter spread trades:* $x_{i,t} = -x_{-i,t}$
- *All risky asset markets clear:* $x_{i,t} + y_{i,t} = 0.$
- *Risk premia are opposites ($\Rightarrow$ price wedge)*

$$\phi_{i,t} = -\phi_{-i,t} \qquad \Rightarrow \qquad \phi_{i,t} = \frac{p_{-i,t} - p_{i,t}}{2}$$

Risk premium:

$$\phi_{i,t} \quad \equiv \quad \underbrace{\frac{D_i}{r}}_{\substack{\text{Price w/o} \\ \text{dividend risk}}} \quad - \quad p_{i,t}$$

## Roadmap

- Model.
- **Riskfree arbitrage.**
- Risky arbitrage.

# RISKFREE ARBITRAGE

- *No fundamental risk:*
  - Assets $i$ and $-i$ pay identical dividends ($\sigma_i^f = 0$).

$$
\begin{aligned}
dD_{i,t} &= D_i dt + \sigma_i dB_{i,t} \\
dD_{-i,t} &= D_i dt + \sigma_i dB_{i,t}
\end{aligned}
$$

- *No supply risk:*
  - Constant $u_{i,t}$: $u_{i,t} = u_i$
  - Define $\mathcal{A} \equiv \{i \in \mathcal{I} : u_i > 0\}$

- Symmetric equilibrium with $\phi_{i,t}$, $x_{i,t}$ and $W_t$ deterministic.

## Arbitrageurs

- Dynamic budget constraint

$$dW_t = \Bigg[ \underbrace{-\beta W_t}_{\substack{\text{consumption} \\ \text{(log utility)}}} + \underbrace{rW_t}_{\substack{\text{riskfree} \\ \text{return}}}$$

$$+ \sum_{i \in \mathcal{I}} x_{i,t} \underbrace{\left( D_i + \frac{dp_{i,t}}{dt} - rp_{i,t} \right)}_{\substack{\equiv \Phi_{i,t} \\ \text{expected excess return per leg}}} + \underbrace{\sum_{i \in \mathcal{A}} \left( x_{i,t} + x_{-i,t} \right) \sigma_i \frac{dB_{i,t}}{dt}}_{\substack{= 0 \\ \text{no dividend risk}}} \Bigg] dt$$

- By symmetry:

$$\boxed{dW_t = \left[ -\left(\beta - r\right) W_t + 2 \sum_{i \in \mathcal{A}} x_{i,t} \Phi_{i,t} \right] dt}$$

**Proposition 1:** *Each arb maxes out his constraint with trades* $(i, -i)$ *s.t.*

$$i \in \arg\max_{j \in \mathcal{A}} \left( \frac{\Phi_{j,t}}{m_j} \right)$$

Intuition:

- Arbitrageurs face riskfree opportunities.
- $\Rightarrow$ Seek the highest "excess return on collateral".

## $(i,t)$-**investors**

$$\max_{y_{i,t}} \quad \underbrace{y_{i,t}\Phi_{i,t}}_{\text{expected excess return}} \quad - \quad \underbrace{\frac{a_i\sigma_i^2}{2} \cdot (u_i + y_{i,t})^2}_{\text{cost of risk}}$$

- FOC:

$$\Phi_{i,t} = a_i \cdot \sigma_i^2 \cdot (u_i + y_{i,t})$$

- Market clearing ($y_{i,t} = -x_{i,t}$) $\Rightarrow$

$$\boxed{\Phi_{i,t} = a_i \cdot \sigma_i^2 \cdot (u_i - x_{i,t})}$$

## Equilibrium

- Financial constraint:

$$\boxed{\sum_{i \in \mathcal{I}} m_i |x_{i,t}| \leq W_t}$$

- FOC:

$$\boxed{\Phi_{i,t} = a_i \cdot \sigma_i^2 \cdot (u_i - x_{i,t})}$$

- Dynamic budget constraint:

$$\boxed{dW_t = \left[ -(\beta - r)\, W_t + 2 \sum_{i \in \mathcal{A}} x_{i,t} \Phi_{i,t} \right] dt}$$

**Corollary 1:** *All opportunities yield the same return on collateral*

$$\exists \Pi_t \in [0,1), \quad \forall i, \qquad \frac{\Phi_{i,t}}{m_i} = \Pi_t$$

Intuition:

- Arbitrageurs seek the highest return on collateral.
- $\Rightarrow$ Equalization in equilibrium.

## Arbitrage Capital Dynamics

- *Important property: Low arbitrage capital $\Rightarrow$ High excess returns.*
  - Financial constraint + FOC:

$$\sum_{i \in \mathcal{I}} m_i |x_{i,t}| \leq W_t \qquad \text{and} \qquad \Phi_{i,t} = a_i \cdot \sigma_i^2 \cdot (u_i - x_{i,t})$$

  - $W_t$ small $\Rightarrow$ $|x_{i,t}|$ small $\Rightarrow$ $\Phi_{i,t}$ large.

**Lemma 2:**

- *Dynamics:* $\qquad dW_t = [\Pi_t - (\beta - r)] \cdot W_t \cdot dt$
- *If $W_t \geq W_\infty$: Arbitrage capital $W_t$ decreases monotonically towards $W_\infty$.*
- *If $W_t \leq W_\infty$: Arbitrage capital $W_t$ increases monotonically towards $W_\infty$.*

## Selected Implications

- Closed form for arbitrageurs' capital $W_t$, positions $x_{i,t}$ and leverage, arbitrage profitability $\Phi_{i,t}$, asset risk-premia $\phi_{i,t} \Rightarrow$ Cross-section and time-series properties.

**Corollary 4:** *Higher margins $\Rightarrow$ Larger premia (more illiquidity)*

$$m_i > m_j \qquad \Rightarrow \qquad \phi_{i,t} > \phi_{j,t}$$

**Corollary 5:** *Higher margins $\Rightarrow$ Premia are more sensitive to supply and arb capital*

$$m_i > m_j \qquad \Rightarrow \qquad \frac{\partial \phi_{i,t}}{\partial u_{k,t}} > \frac{\partial \phi_{j,t}}{\partial u_{k,t}} \quad \text{and} \quad \left| \frac{\partial \phi_{i,t}}{\partial W_t} \right| > \left| \frac{\partial \phi_{j,t}}{\partial W_t} \right|$$

Intuition:

- Equalization of returns on collateral.
- More collateral $\Rightarrow$ Arbs demand higher excess returns $\Phi$.
- Risk premia $\phi =$ PV of future excess returns $\Phi$.

## Roadmap

- Model.
- Riskfree arbitrage.
- **Risky arbitrage.**

# RISKY ARBITRAGE

- *Fundamental risk*:
  - Assets $i$ and $-i$ pay different dividends ($\sigma_i^f \neq 0$).

$$dD_{i,t} = D_i dt + \sigma_i dB_{i,t} + \sigma_i^f dB_{i,t}^f$$

$$dD_{-i,t} = D_i dt + \sigma_i dB_{i,t} - \sigma_i^f dB_{i,t}^f$$

- *Supply risk*:
  - Shocks $u_{i,t}$ are stochastic:

$$du_{i,t} = \kappa_i^u \left(u_i - u_{i,t}\right) dt + \sigma_i^u \, dB_{i,t}^u$$

- No closed form $\Rightarrow$ Study near the riskfree case:
  - Arbitrage risk is small: $\sigma_i^f \simeq 0$ and $\sigma_i^u \simeq 0$.
  - Supply is slowly mean-reverting: $\kappa_i^u \simeq 0$.

## Contagion / Liquidity Linkage

**Lemma 3:** *Fundamental/supply shocks in one market affect all asset prices.*

<u>Mechanics:</u>

- Fundamental shock:
  - Indirect effect: Arbitrage Capital $\Rightarrow$ Asset Prices $\Rightarrow$ Arbitrage Capital $\Rightarrow$ Etc.
- Supply shock:
  - Indirect effect $+$ Direct effect

<u>Intuition:</u>

- Arbitrage capital $W_t$ affects all asset prices.

**Lemma:** *The volatility of arbitrage capital is $\cap$-shaped in arbitrage capital.*

Intuition:

- Two drivers of the volatility of total arbitrage capital (not per dollar).

  Price Volatility $\times$ Exposure (positions)

- A capital increase affects both in opposite ways.
  - "Dampening": Supply shocks affect premia less $\Rightarrow$ Capital is less volatile.
  - "Exposure": Larger positions $\Rightarrow$ Capital is more volatile.

- When capital is low, the *exposure effect* is large:
  - Binding constraint $\Rightarrow$ More wealth implies larger positions $\Rightarrow$ Higher exposure.
  - Moreover $W$ small $\Rightarrow$ High returns $\Rightarrow$ Higher volatility.
  - $\Rightarrow$ Exposure effect dominates $\Rightarrow$ Capital is more volatile.

- Opposite when capital is low.

# Volatilities

**Proposition 7:** *Premia volatility (due to supply shocks) is $\cap$-shaped in arb capital.*

Intuition:

- Arbitrage capital affects premia.
- Its volatility affect premia's volatility.

## Correlations

**Proposition 8:** *Consider* $(i, j) \in \mathcal{A} \times \mathcal{A}$, $i \neq j$.

- *Correlation (due to supply shocks) is* $\cap$*-shaped in arb capital.*
- *Correlation between* $i$ *and* $-i$ *is U-shaped.*

Intuition:

- Arbitrage capital affects asset prices.

- Assets $(i, j) \in \mathcal{A} \times \mathcal{A} \Rightarrow$ Both have positive correlation with arbitrage capital.
- $\Rightarrow$ Volatility of capital increases their correlation.

- Assets $(i, -i) \in \mathcal{A} \times \mathcal{B} \Rightarrow$ Opposite correlations with arbitrage capital.
- $\Rightarrow$ Volatility of capital decreases their correlation.

## Liquidity $\equiv$ 1 / (Price impact of supply shocks)

**Proposition 6:** *Liquidity is U-shaped in arbitrage capital.*

Intuition:

- Direct effect: Decreases with arbitrage capital.
- Indirect effect: $\cap$-shaped.

# RESEARCH AGENDA

- Applications + Extensions:
  - Relation to standard models with incomplete markets, transaction costs, etc.
  - Diversification vs. Contagion.
  - Mobility of arbitrage capital.

- Endogenous constraints:
  - Information asymmetry? Moral hazard?
  - Technical: Optimal contract in a dynamic principal-agent model... in GE.

- Welfare:
  - Equilibrium is not constrained efficient (Gromb and Vayanos 2002).
  - $\Rightarrow$ Policy.