Risk Sharing and Contagion in Networks

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Introduction: Motivation

- Recent events made clear we cannot look at financial institutions in isolation from others

- In this paper we aim to investigate the effects of different forms of interconnections among (financial) firms,
  in particular for the capacity of the system to withstand shocks.
  Will determine the optimal pattern of connections
  and study whether this can be sustained in equilibrium.

- *Key trade-off*: More interconnections
  $\Rightarrow$ higher levels of insurance, but also
  $\Rightarrow$ higher risk of contagion (large shocks can generate widespread default in the system)

Introduction: Main Features of the Model

- Network with nodes = financial firms:
  - Presence of a (direct or indirect) link among two firms means they are in a situation of mutual (direct or indirect) exposure
  - Degree of exposure of a firm to another firm: depends on number of linkages of the two firms and the distance between them.

- When a random shock negatively affects the income/cash available of a firm:
  - all firms directly or indirectly linked to the firm hit must bear part of the shock, in proportion to their exposure to the firm
  - firms unable to make required payments must default, and this is costly

Introduction: Main Features of the Model (II)

- To analyze key trade-off, we compare performance of network structures that differ in two main dimensions:
    1. Degree of *segmentation* (number of disjoint components in which system is divided).
    2. *Tightness* of connections (extent to which firms are directly or indirectly linked among them)

- Address two main questions:
    1. How does the optimal network structure vary with the stochastic structure of shocks?
    2. What is the relationship between optimality and incentives of individual firms (stability)?

Introduction: Main Results

- *Trade-off insurance/contagion* clearly emerges:

  ia) Optimal segmentation maximal when shock distribution exhibits fat tails, minimal with thin tails;
  ib) Intermediate levels of segmentation optimal with sufficient probability mass both on large and small shocks

  ii) Maximal connectivity within a component not always optimal: sparser structures (rings) optimal for some distributions, they avoid extreme outcomes where either all stand or all default in a component.

  iii) With firms of different sizes: asymmetric structures (stars) may help to prevent contagion.

- Potential conflict Equilibrium/Optimality:
  Equilibrium typically requires asymmetry in size of components, not optimal

Literature

Two main strands:

1. Detailed microfoundation of linkages, simple network and shock structures:

Allen and Gale (2000)
Lagunoff and Schreft (2001)
Leitner (2005)
Allen, Babus, Carletti (2011), ...

Optimality and pairwise stability: Bramoullé Kranton (2007)

2. Random (large) networks, simple contagion mechanics

Nier et al. (2007)
Blume et al. (2011), ...

The Environment

- N financial firms, ex ante identical (for now)
- At any point in time each firm has an investment opportunity (assets), to be financed with deposits.
- Gross return on investment is a random variable $\widetilde{R} = R - \widetilde{L}$, s.t.:
  - with prob. $1 - \pi_s - \pi_b$, normal return: $\widetilde{R} = R$
  - with prob. π_s, a *small* shock s hits: $\widetilde{R} = R - L_s$
  - with prob. π_b, a *big* (random) shock b hits: $\widetilde{R} = R - \widetilde{L}_b$, with $\widetilde{L}_b \geq L_s$
- Gross return due on deposits: M, to ensure expected return r

Assumption 1: $R(1 - \pi_s - \pi_b) > r$ and $R - L_s < r$

The Environment (II)

- A.1 $\Rightarrow$ Project is viable, but if firm relies on its own resources, can't repay depositors and hence must **default** both if a shock s or a shock b hits it.
  Default entails a significant cost for firm (loss of all future earnings possibilities)

- Each firm can reduce default risk by establishing **linkages** with one or more other firms:
  exchange a fraction $1-\alpha$ of its assets with (the same amount of) assets of one or more other firms.

  Exchange can be direct or indirect (through rounds of *securitization*): firm's return becomes a weighted average of return of own and partners' projects.

Assumption 2:

i. $\alpha \geq \frac{1}{2}$ (moral hazard);
ii. Shocks are rare, *at most* one firm is hit by a shock at any date
iii. Connected firm never default if directly (or indirectly) hit by small s shock when $\alpha = 1/2$ (risk sharing):

$$\frac{1}{2}(R - L_s) + \frac{1}{2}R - M \geq 0$$

iv. A connected firm always defaults if one of the directly linked firms is hit by largest of the b shocks (contagion: default cannot be avoided with maximal connectivity):

$$R(1 - \frac{2}{N-1}) \leq M$$

v. s shocks are significantly more likely than b ones: $\pi_s > N\pi_b$

Comparing Network Structures

- **Question:** What is the pattern of linkages (of asset exchanges) that allows to maximize welfare (that is, minimize the probability of default)?

  - Under A.2iii.,v. autarky never optimal.

  - Set $\alpha = 1/2$: Different network structures have different implications only for the ability of a firm to *survive when a large b shock hits another firm* to whom it is connected.

- **Note:** mutual exposure/risk of contagion comes from firms' cross ownership of each other's project.
  Default of a firm has no direct implication for solvency of other firms (unlike case where mutual exposure comes from mutual loans between firms).

Network Structures: segmentation

Will consider symmetric structures that differ in two main dimensions:

1. **Degree of segmentation**:
   number C of disjoint components
   (each with $K + 1 \equiv \frac{N}{C}$ firms directly or indirectly linked among them).

   The larger the segmentation (the smaller K), the fewer the firms affected by a shock but the larger their mutual exposure and hence the probability they default if a b shock hits their component.

   - $K = N - 1$: fully connected system, has the best ability to withstand shocks. But, in the event of sufficiently large shocks $\widetilde{L}_b$, could have generalized default in the system.
   - $K = 1$: maximal insulation from shocks

Network Structures: density.

2. **Tightness of internal connections:**
fraction of indirect vs. direct linkages within each component. Focus on two polar cases:

i) *Completely connected components:* only direct linkages

In a component of size $K + 1$ each firm ends up with a fraction α of its original assets and $(1 - \alpha)/K$ of assets of each of the other firms.
It **defaults** when:

$R - M < \alpha L_b$ for shocks hitting it directly,

$R - M < \frac{1-\alpha}{K} L_b$ for shocks hitting any other firm in its component

Modelling Ring Structures

ii) *Minimally connected components (rings)*:

- Each firm is directly linked with two firms. Indirect linkages are formed by iterating $\frac{K}{2}$ times the exchange of - then composite - assets with the two 'neighboring' firms (rounds of securitization).

  Pattern of exposure to returns of projects of firms in component described by matrix

$$A = \begin{bmatrix} \theta & (1-\theta)/2 & 0 & \cdots & (1-\theta)/2 \\ (1-\theta)/2 & \theta & (1-\theta)/2 & 0 & \cdots \\ 0 & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ (1-\theta)/2 & 0 & \cdots & (1-\theta)/2 & \theta \end{bmatrix}^{K/2}$$

Modelling Ring Structures (II)

- Properties of matrix A :
  - $a_{ii} = \alpha$
  - $a_{ij} \geq a_{il}$ when $|i - j| \geq |i - l|$: exposure falls with distance
  - $a_{ij} > 0$ for all i, j: everybody connected within component

- **A continuum approximation:**
  will approximate above pattern of exposure with:
  $f^c(d) \to \mathbb{R}_+$, fraction of the shock hitting another firm at distance $d \in [0, K/2]$ which has to be borne by a firm
  - $f^c(\cdot)$ decreasing, continuous, piecewise linear, such that
    - $f^c(0) = \alpha; \qquad f^c(K/2) = 0,$
    - $2 \int_0^{K/2} f^c(x)\, \mathrm{d}x = 1 - \alpha$

Modelling Ring Structures (III)

- A firm in a ring **defaults** when:

$$R - M < \alpha L_b \text{ for shocks hitting firm directly,}$$

$$R - M < f^c(d) L_b \quad \text{for shocks hitting other firms in component at distance } d$$

In what follows:

- Normalize, w.l.o.g., $R - M = 1$

  Objective: determine the optimal degree of segmentation K and tightness of internal connections that minimizes the expected probability of default of a firm for different properties of the distribution $\Phi(L_b)$.

Optimal Financial Structures: thin/fat tails distributions

Proposition

Let the shock $\widetilde{L}_b$ be Pareto distributed on $[1, \infty)$ with $\Phi(L_b) = 1 - 1/L_b^{\gamma}$.
- The optimal degree of segmentation both for the ring and the complete structures is
- maximal ($K^* = 1$) if $0 < \gamma < 1$
- minimal ($K^* = N - 1$) if $\gamma > 1$

- Complete dominate ring structures for all γ.

When distribution of shocks exhibits fat tails, defaults minimized by minimizing linkages.
With thin tails, optimal to have a single connected component.

Argument

- For the ring, expected default rate is

$$
\begin{aligned}
D_r(K,\gamma) \;=\; & 2\left[\left(\frac{K}{2}\left(\frac{1}{\gamma+1}\right)+\left(\frac{1}{K+1}\frac{\gamma}{\gamma+1}\right)\right.\right. \\
& \left.\left.-\frac{2}{K-1}\left(\frac{1}{2}-\frac{\gamma}{\gamma+1}\frac{1}{K+1}\right)\right)\right] \\
& \times\left(\frac{1}{K+1}\right)^{\gamma}+2\left[\frac{2}{K-1}\frac{(1/2)^{\gamma+1}}{(\gamma+1)}\right]
\end{aligned}
$$

  monotonically increasing (decreasing) in K when $\gamma < 1$ ($\gamma > 1$)

- Same is true for the completely connected structure, for which expected default rate is

$$
D_c(K,\gamma) = K\left(\frac{1}{2K}\right)^{\gamma}
$$

Optimal Financial Structures: mixed distributions

Proposition

Let $\Phi(L_b)$ be a mixture of two Pareto distributions with $\gamma > 1$ and $\gamma' < 1$, with weights p and $1 - p$. Then the optimal degree of segmentation for the complete structure obtains at $1 < K^* < N - 1$ (for $p_1 < p < p_2$)

- Optimum at solution of

$$\begin{array}{c} \min_{K_i,n} \sum_{i=1}^{n} \frac{K_i+1}{N} D_c(K_i) \\ \text{s.t. } \sum_{i=1}^{n} \frac{K_i+1}{N} = 1 \end{array}$$

obtains at $K_i = K^*$ for all n (symmetric, by convexity of $D_c(K_i)$ in relevant range)

- For ring structures numerical analysis yields similar results, though optimal size of components is larger.
- Complete structure still better than ring

An example

Let $\gamma = 2$, $\gamma' = 0.5$ and $p = 0.95$.

- optimal degree of segmentation for complete structure obtains at $K^* = 5.65$, with $D_c^* = 0.13$.
- for the ring structure it obtains at $K^* = 8.02$, with $D_r^* = 0.145$.

Optimality of minimally connected components

Proposition

Let $\Phi(L_b)$ be a mixture of a Pareto distribution with $\gamma \in (1,2)$, with weight p, and a discrete distribution with all mass on $\bar{L}_b = 2(N-1)+1$. Then, if N and p are such that

$$N > 1 + \left(\frac{1}{4^{\gamma-1}} - \frac{1}{5^{\gamma}} + \frac{\frac{1}{2^{\gamma-1}} \frac{1}{(\gamma+1)}}{\frac{\gamma+1}{2^{\gamma+1}} - \frac{1}{\gamma+1}} \right)^{\frac{1}{2-\gamma}}$$

$$\frac{(1-p)}{p} < (\gamma-1) \left(\frac{1}{2(N-1)} \right)^{\gamma}$$

the optimal financial structure is a single ring component.

- Pareto distribution has thin tails $\Rightarrow$ components large.
- But with low probability $(1-p)$ big shock $\bar{L}_b$ hits, causing default of everybody in single complete structure, while with sparser connections (ring) some firms survive.

Stability and optimality

Is optimality consistent with firms' incentives to establish linkages?

- Individual incentives captured by notion of *Coalition Proof Equilibrium* (CPE): No subset of positive measure and a strategy profile that makes them improve.

Proposition

Let $\Phi(L_b)$ be a mixture of two Pareto distributions with $\gamma > 1$ and $\gamma' < 1$ such that the optimal structure is completely connected with $1 < \hat{K} < N - 1$.
Generically the optimal structure is not supportable as a CPE.

- Optimality requires all components of same size (Prop. 2)
- CPE requires all but one component to have size $\hat{K} \in \arg\min D_c(K)$, that is minimizing expected defaults of the component (generically $\neq K^*$) and one, strictly smaller component.

Stability and optimality (II)

More specifically:

- A CPE structure:
  - cannot have any complete component with $K > \hat{K}$:
    a subset would want to delete some links (get smaller).
  - cannot have more than one component with $K < \hat{K}$:
    a subset in the first component would benefit by deleting links and forming links with the second component.

- Let $Q = int\left(\frac{N}{\hat{K}+1}\right)$. **Only CPE structure** has:
  - Q (complete) components of size $\hat{K} + 1$.
  - one (complete) component of size $N - Q(\hat{K} + 1)$ (the remainder).

  Members of $\hat{K}$ do not want to change. "Outsiders" not accepted.

Asymmetric Firms I

- For this section structures are completely connected.
- N firms, partitioned in subsets $N_1, ...N_n$, so $N = \cup_{l=1}^{n} N_n$, and if $j \in N_l$ a big shock follows F_{N_l}.
- $D\left(K, F_{N_l}\right)$ is expected number of defaults, if a big shock hits a firm $j \in N_l$ in complete component of size $K + 1$.
- Each firm $j \in G_i$, whose size is K_i so that $N = \cup_{i=1}^{I} G_i$.
- Expected number of deaths if $j \in N_l$ is hit $KP\left(\frac{(1-\alpha)L}{K} > 1\right)$ this expression only depends on K and the distribution of L.
- Define $K_{N_l}^{*}$ as the minimizer in K of $D\left(K, F_{N_l}\right)$.

Proposition

Suppose that $|N_l|$ is a multiple of $K_{N_l}^{*} + 1$ for every l. The optimal configuration of N is such that all firms are in groups with firms of the same type, and of size $K_{N_l}^{*} + 1$ for firms of type N_l.

Asymmetric Firms II

- We now assume return on firm j when no shock is βR and probability of a shock of size βL is equal to probability of shock size L for size 1 firm.
- A size β firm exchanges assets with size 1 in proportion $1/\beta$ to 1.
- Same structure as before in terms of shocks.
- N_l^1 set of type l and size 1, firms, N_l^β type l and size *beta* firms.
- Let component G_i of K_1 size 1 and K_β size β firms, where each firm $j \in N_l$. Let $K = K_1 + \beta K_\beta$.
- Define $K_{N_l}^*$ as the minimizer in K of $D\left(K, F_{N_l}\right)$.

Proposition

Suppose that $\left|N_l^1\right| + \beta\left|N_l^\beta\right|$ *is a multiple of* $K_{N_l}^* + 1$ *for every* l. *The optimal configuration of N is such that all firms are in groups with firms of the same type* l*, and of size* $K_{N_l}^* + 1$.

Conclusions

- Have considered a very stylized model to study trade-off of forming linkages:
  benefits of risk sharing vs. costs of contagion in a context with small probability, random shocks.
- If shocks are typically large (small), the optimal configuration exhibits maximal (minimal) segmentation in complete components.
- For richer shock patterns, the optimal configuration may involve intermediate levels of segmentation and/or sparse connectivity.
- If firms are asymmetric in size or type of shock, assortativity in type of shock.
- Optimality and stability are typically in conflict: in equilibrium we have both too large and too small components.