Debt and Risk Sharing in Incomplete Financial Markets: A Case for Nominal GDP Targeting*

Kevin D. Sheedy†
London School of Economics

First draft: 7th February 2012
This draft: 1st June 2012

Abstract

Financial markets are incomplete, thus for many agents borrowing is possible only by accepting a financial contract that specifies a fixed repayment. However, the future income that will repay this debt is uncertain, so risk can be inefficiently distributed. This paper argues that a monetary policy of nominal GDP targeting can improve the functioning of incomplete financial markets when incomplete contracts are written in terms of money. By insulating agents' nominal incomes from aggregate real shocks, this policy effectively completes the market by stabilizing the ratio of debt to income. The paper argues that the objective of nominal GDP should receive significant weight even in an environment with other frictions that have been used to justify a policy of strict inflation targeting.

JEL classifications: E21; E31; E44; E52.

Keywords: incomplete markets; heterogeneous agents; risk sharing; nominal GDP targeting.

*This version is preliminary and incomplete. I thank Wouter den Haan, Albert Marcet, and Matthias Paustian for helpful discussions, and participants at the Centre for Economic Performance annual conference for their comments.

†Department of Economics, London School of Economics and Political Science, Houghton Street, London, WC2A 2AE, UK. Tel: +44 207 107 5022, Fax: +44 207 955 6592, Email: k.d.sheedy@lse.ac.uk, Website: http://personal.lse.ac.uk/sheedy.

1 Introduction

In the last twenty years, inflation targeting has been adopted by an increasingly large number of central banks. Accompanying that rise has been a debate on whether inflation targeting has narrowed the focus of monetary policy too much, in particular, whether it leads central banks to take their eyes off developments in financial markets. This debate has understandably become more active in recent years. Some would argue that too much focus on low and stable inflation may allow potentially dangerous financial imbalances to build up in credit and asset markets (Christiano, Motto and Rostagno, 2007, Christiano, Ilut, Motto and Rostagno, 2010, White, 2009), perhaps justifying a monetary policy that leans against large movements in asset prices and credit growth. Others would argue that targeting asset prices is either undesirable or ineffectual (Bernanke and Gertler, 2001), or that 'flexible' inflation targeting can be adapted to accommodate a world where there is a tension between price stability and financial stability, and that it would be unwise to discard inflation targeting's role in providing a clear nominal anchor (Woodford, 2011).

The canonical justification for inflation targeting (see, for example, Woodford, 2003) is that it promotes efficiency in goods markets by avoiding the relative-price distortions that would occur with high or volatile inflation. Stability of inflation can also contribute to a efficient level of overall economic activity (closing the 'output gap') because the Phillips curve implies a link between inflation fluctuations and output gap fluctuations. The theories of nominal rigidities that provide the intellectual support for this view have been extensively researched. Still much less researched, however, are the types of monetary policies that would promote efficiency in financial markets. Here, there is less consensus on the distortions that monetary policy should be aiming to avoid or to correct, and the extent to which correcting financial-market distortions is in conflict with correcting distortions in goods markets.

This paper begins from the observation that financial markets are incomplete, so many agents wishing to borrow can only do so by agreeing to financial contracts that specify fixed repayments in the future. The future real incomes that will be used to repay these debts are however uncertain. Such incomplete contracts lead to the burden of risk being unevenly distributed among agents, which is inefficient. If financial contracts are written in terms of monetary units then a monetary policy that targets nominal GDP can help in minimizing the inefficiency of financial markets. The argument is that this policy stabilizes the debt-to-GDP ratio in response to aggregate shocks, which will effectively lead to financial markets that completely span the risk coming from aggregate shocks. A policy of strict inflation targeting would fail to deliver this efficient outcome and would instead lead to inefficient fluctuations in levels of debt and interest rates.

These conclusions are obtained analytically from a tractable heterogeneous-agents model that can be used to study the effects of different monetary policies on the efficiency of financial markets and to derive optimal monetary policy. The model contains overlapping generations of individuals who earn uncertain real incomes that vary over each individual's lifetime. Individuals are differently exposed to shocks to GDP because of the pattern of labour income over the lifecycle: most labour income is earned when middle-aged, and relatively little when young or old. There is thus a role for financial markets in facilitating borrowing, saving, and risk sharing between individuals. Individuals with concave utility functions would like to smooth consumption over time and over different states of the world.

The financial markets that individuals have access to are incomplete. It is assumed individuals cannot borrow by short-selling state-contingent bonds (Arrow-Debreu securities). Borrowing must take the form of non-contingent debt contracts, which are denominated in terms of money. These are arguably reasonable assumptions for most households. As savers, individuals can buy any assets that are traded. In equilibrium, savers must hold the securities issued by borrowers, so the only financial market that operates is for nominal non-contingent bonds.

In this environment, individuals borrow when young and repay their debts when middle-aged. They then save by lending to the next generation of young to fund consumption when they are old. However, when the young choose how much to borrow, they do not know the future course the economy will take, which will affect their future labour income. Will there be a productivity slowdown, a deep and long-lasting recession, or even a 'lost decade' of poor economic performance to come? Or will unforeseen technological developments or terms-of-trade movements boost future incomes and good economic management successfully steer the economy on a path of steady growth? The young do not know what is to come, but must choose how much to borrow (and fix what they must contractually repay) prior to this information being revealed.

Now suppose the central bank is following a policy of strict inflation targeting. Such a policy transforms financial contracts specifying non-contingent *nominal* repayments into contracts requiring non-contingent *real* repayments. The real value of the debts the young must repay when middle-aged remains the same irrespective of what subsequently happens to the economy. Suppose that GDP is much lower than the currently middle-aged expected when they were young. Their labour income turns out to be too low relative to what they must repay, hence their consumption (whether now or in the future) must fall sharply. At the same time, the savings of the old maintain their real value, thus their consumption is largely insulated from the shock because it is principally financed from savings rather than labour income. Similarly, had the realization of GDP growth been better than expected then the middle-aged would have benefited disproportionately because their debt repayments are fixed, while the old would 'miss the party' and be stuck with a relatively low consumption level.

This model is designed as a simple representation of a world where some agents are leveraged, while other agents hold the corresponding debt obligations as safe assets. The young would like to smooth consumption, but can only do this by leveraging their future income stream. Consumption, as the residual between fluctuating income and fixed debt repayments, is more volatile as a result of this leverage. In equilibrium, if borrowers cannot issue risky liabilities, savers end up holding risk-free assets. Risk is thus very unevenly distributed among agents.

The consequences of this uneven distribution of risk will be seen in financial markets where credit aggregates, interest rates, and asset prices will display larger fluctuations in response to shocks were the central bank to follow a policy of strict inflation targeting. Following a positive shock to GDP growth, inflation targeting maintains the real value of maturing debt, so the ratio of repayments to income falls. This sets off a credit boom with new lending growing faster than GDP. The intuition is that with risk concentrated in a subset of the population, the beneficiaries of the shock lend to those who did not directly benefit, with the aim of smoothing consumption over their lifetimes. In order to persuade others to borrow more, this expansion of credit drives down long-term real interest rates, inflating the prices of bonds and other assets by increasing the present discounted value of any stream of future payments. Similarly, strict adherence to an inflation target following an adverse shock to GDP leads to a greater deleveraging as the burden of repaying maturing debt rises, which reduces funds available for new lending. As credit becomes scarce, real interest rates rise and asset prices fall.

In what sense are the outcomes described above inefficient? The young realize that a non-contingent debt contract, while allowing them to consume more when their income is initially low, exposes them to greater consumption volatility in the future. Ex post, they would have large gains in some states of the world, and large losses in other states, while the old would have little to gain or lose in any states of the world. This is inefficient from an ex-ante perspective. If financial markets were complete, the young would prefer to borrow by short-selling contingent bonds. They would want to sell relatively few bonds paying off in future states of the world where GDP and thus incomes are low, and sell relatively more in good states of the world. As a result, prices of contingent bonds paying off in bad states would be relatively expensive and those paying off in good states relatively cheap. These price differences would entice middle-aged individuals saving for retirement to shift away from non-contingent bonds and take on some risk in their portfolios. In this way, mutually welfare-improving trades would take place in complete financial markets.

Through this mechanism, complete financial markets would allow for risk sharing among the individuals in the economy. This provides insurance for the young by reducing their future liabilities in states of the world where income is low, balanced by increasing liabilities in good states. With risk evenly distributed among individuals, credit volumes and asset prices would be more stable. The consequences of a shock to GDP would not fall on some individuals more than others because maturing liabilities would move in line with incomes. With no disproportionate winners or losers, there would be no fluctuations in available funds to lend relative to the demand to borrow those funds. The economy then enjoys a more stable debt-to-GDP ratio, and more stable asset prices.

In the absence of complete financial markets, strict inflation targeting implies that risk is not shared efficiently among individuals, and that quantities and prices in financial markets are too volatile as a result. But what alternative monetary policies could do better? While incomplete financial contracts are non-contingent in *nominal* terms, the real value of the monetary unit depends on monetary policy, so the *real* contingency of financial contracts is actually endogenous to monetary policy. Efficient risk sharing requires that individuals subject to income risk are insured through adjustment of liabilities in line with income. This suggests a monetary policy of targeting the debt-to-GDP ratio. Since the non-contingent terms of financial contracts are expressed in terms of money, monetary policy can influence this ratio simply by changing the price level.

Faced with aggregate shocks to real GDP, it is shown that stabilizing the debt-to-GDP ratio at some target value in a world of incomplete markets is equivalent to achieving the same allocation of consumption as would prevail with complete financial markets. Thus, a monetary policy that achieves the ex-ante efficient sharing of risk must stabilize this ratio. However, simply adopting the debt-to-GDP ratio as a target for monetary policy in place of targeting a rate of inflation is highly unsatisfactory. While one instrument of policy is sufficient to hit one target, a policy of targeting a ratio completely fails to provide the economy with a nominal anchor. It would leave inflation expectations undetermined, which could then drift arbitrarily. This abandons one of the most important advantages of inflation targeting: the provision of a clear nominal anchor.

However, there exists an alternative policy that both achieves Pareto efficiency in financial markets and delivers a nominal anchor. Given that financial contracts specify liabilities as fixed monetary amounts, stabilization of the debt-to-GDP ratio is equivalent to stabilizing the *nominal* value of income, in other words, a policy of targeting nominal GDP. Even if the real value of GDP fluctuates unexpectedly, borrowers are shielded from excessive risk by a monetary policy that ensures nominal incomes do not deviate from predetermined nominal liabilities. Furthermore, by setting a target for the nominal level of GDP, given real GDP, the price level and other nominal variables are firmly anchored. Thus, a policy of nominal GDP targeting effectively completes financial markets in regard to aggregate shocks (though obviously not for idiosyncratic shocks) as well as pinning down expectations of inflation.

Such a policy naturally entails some unpredictable fluctuations in inflation. However, these would not be of an excessive magnitude: the standard deviation of inflation would equal the standard deviation of real GDP growth. Furthermore, it is important to appreciate that this inflation is *not* costly for one of the reasons often cited: the redistribution of wealth between debtors to creditors. Some ex-post transfers of wealth resulting from unexpected inflation are precisely what is needed to obtain ex-ante efficient risk sharing. While there are always winners and losers from any transfers ex post, it could be argued that under nominal GDP targeting these simply mimic the contingencies in repayments between debtors and creditors that would have been mutually agreeable in a world of complete financial markets. But nonetheless, it could still be objected that there are other costs of inflation such as relative price distortions in goods markets and menu costs that have been ignored in the argument above.

To address these potential trade-offs, the model is extended to include more frictions. First, introducing differentiated goods and nominal rigidities in goods prices implies that inflation leads to inefficiencies in the allocation of resources as relative prices are distorted. To study the trade-off between efficiency in financial markets and efficiency in goods markets, a utility-based loss function is derived. The loss function contains two terms: inflation squared, representing the losses from relative-price distortions in goods markets, and the debt-to-GDP ratio squared, representing the losses from the incompleteness of financial markets. Optimal monetary policy is then derived analytically. Unsurprisingly, the optimal policy is a mixture of the two policies that address each of the two inefficiencies individually: nominal GDP targeting for financial markets, and strict inflation targeting for goods markets. A mixture of these two policies is a weighted nominal GDP target, with the price level receiving a higher weight than real GDP. For reasonable calibrations of the model, the optimal policy remains far from strict inflation targeting. The intuition is that the inflation fluctuations under nominal GDP targeting are relatively modest given the volatility of real GDP.

A further extension introduces an endogenous labour supply decision. With flexible prices and wages, the optimality of nominal GDP targeting does not depend on whether GDP is modelled as an exogenous endowment, or as the outcome of a labour supply decision in a world with TFP shocks. However, combined with sticky prices, it means that the central bank also needs to be concerned about stabilizing the output gap. It is shown that the utility-based loss function now includes three terms: inflation squared, debt-to-GDP squared, and the output gap squared. The goal of stabilizing inflation is in conflict with the goal of stabilizing the debt-to-GDP ratio as already described. In conventional analyses of optimal monetary policy, there is often a 'divine coincidence' between the policy that stabilizes inflation and the policy that stabilizes the output gap. This is shown to break down with incomplete markets, so there would now be a tension between inflation and output gap stabilization even if the central bank did not care directly about the debt-to-GDP ratio.

This tension between inflation stabilization and output gap stabilization with incomplete markets indicates that the potential inefficiency of strict inflation targeting goes beyond just considerations of risk-sharing. It is shown that the debt-to-GDP ratio acts as an endogenous cost-push shock that drives a wedge between the equilibrium and the efficient levels of aggregate output. The intuition is that with incomplete markets and a policy of strict inflation targeting, labour income risk is unevenly distributed. With stronger income effects for those supplying more labour, this leads to too little labour being supplied when labour productivity is high and too much being supplied when productivity is low. The policy of strict inflation targeting thus reduces the correlation between productivity and hours, implying a sub-optimal response to TFP shocks.

This paper is related to a number of areas of the literatures on monetary policy and financial frictions. First, there is the empirical work of Doepke and Schneider (2006) documenting the effects of inflation in redistributing wealth between debtors and creditors. In the context of government debt, the role of inflation in affecting the real value of nominally non-contingent bonds is the subject of a well-established literature starting from Chari, Christiano and Kehoe (1991), and expanded further in Chari and Kehoe (1999), with Lustig, Sleet and Yeltekin (2008) being a recent contribution. One finding is that inflation needs to be extremely volatile to complete the market. As a result, Schmitt-Grohé and Uribe (2004) and Siu (2004) show that once some nominal rigidity is added to the model so that inflation fluctuations have a cost, the optimal policy is found to be very close to strict inflation targeting. In the context of the household sector, a more closely related contribution is that of Pescatori (2007), who studies the effect of pre-existing inequality on optimal monetary policy decisions in a model with incomplete markets, taking account of the fact that changes in interest rates will affect borrowers and savers differently. Also focusing on households and incomplete markets, and in particular on household saving behaviour, Kryvtsov, Shukayev and Ueberfeldt (2011) study optimal monetary policy in a model where the aggregate level of savings is not socially optimal.

There is now more generally a burgeoning literature on incorporating financial frictions into models for monetary policy analysis. That literature stresses different frictions from those considered in this paper. Cúrdia and Woodford (2009) present a model of credit spreads between savers and borrowers and derive implications for optimal monetary policy. Then there are papers that study borrowing and collateral constraints of the kind found in Kiyotaki and Moore (1997), or the financing of firms subject to costly state verification, as in Bernanke, Gertler and Gilchrist (1999). Contributions in this area include Faia and Monacelli (2007) and De Fiore and Tristani (2009). Work by Christiano, Motto and Rostagno (2010) emphasizes the role played by 'risk shocks' faced by entrepreneurs.

Finally, this paper is related to the existing literature on nominal GDP targeting, with Meade (1978), Bean (1983), and Hall and Mankiw (1994) being key contributions. The argument for nominal GDP targeting here differs from those put forward by other advocates.

The plan of the paper is as follows. Section 2 sets out the basic model. The equilibrium in the case of an endowment economy is derived in section 3, and optimal monetary policy is characterized. Section 4 introduces sticky prices into the model, and section 5 adds an endogenous labour supply decision. Finally, section 6 draws some conclusions.

2 The model

The model is of an economy with overlapping generations of individuals. Time is discrete and is indexed by t. A new generation of individuals is born in each time period and each individual lives for three periods. During their three periods of life, individuals are referred to as the 'young' (y), the 'middle-aged' (m), and the 'old' (o), respectively. An individuals derives utility from consumption of a composite good at each point in his life. At time t, per-person consumption of the young, middle-aged, and old is denoted by $C_{\mathrm{y},t}$, $C_{\mathrm{m},t}$, and $C_{\mathrm{o},t}$. Individuals have identical utility functions, which have a power utility functional form. Future utility is discounted at a rate ϱ ($0 < \varrho < \infty$). The intertemporal elasticity of substitution is σ ($0 < \sigma < \infty$, the coefficient of relative risk aversion is $1/\sigma$). The utility $\mathscr{U}_t$ of the generation born at time t is

$$\mathscr{U}_t = \frac{C_{\mathrm{y},t}^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} + \beta \frac{C_{\mathrm{m},t+1}^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} + \beta^2 \frac{C_{\mathrm{o},t+2}^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}}, \quad \text{where} \quad \beta \equiv \frac{1}{1+\varrho}. \tag{2.1}$$

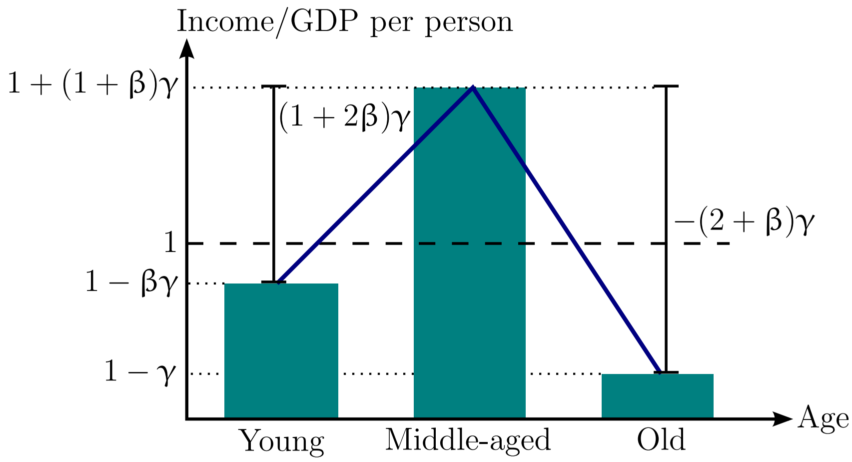
The discount factor β satisfies $0 < \beta < 1$. There is no intergenerational altruism.

The number of young individuals born in any time period is exactly equal to the number of old individuals alive in the previous period who now die. The economy thus has no population growth and a balanced age structure. Assume that the population of individuals currently alive has measure one, with each generation having measure one third. Aggregate consumption at time t is denoted by C_t:

$$C_t = \frac{1}{3}C_{\mathrm{y},t} + \frac{1}{3}C_{\mathrm{m},t} + \frac{1}{3}C_{\mathrm{o},t}. \tag{2.2}$$

All individuals of the same age at the same time receive the same income, with $Y_{\mathrm{y},t}$, $Y_{\mathrm{m},t}$, and $Y_{\mathrm{o},t}$ denoting the per-person incomes (in terms of the composite good) of the young, middle-aged, and old, respectively, at time t. Incomes will be modelled initially as exogenous endowments, with this assumption being relaxed in section 4 and section 5. Age-specific incomes at time t are assumed

Figure 1: *Age profile of non-financial income*



to be time-invariant multiples of aggregate income Y_t, with α_y, α_m, and α_o denoting the multiples for the young, middle-aged, and old, respectively:

$$Y_{y,t} = \alpha_y Y_t, \quad Y_{m,t} = \alpha_m Y_t, \quad Y_{o,t} = \alpha_o Y_t, \quad \text{where } \alpha_y, \alpha_m, \alpha_o \in (0, 3) \text{ and } \frac{1}{3}\alpha_y + \frac{1}{3}\alpha_m + \frac{1}{3}\alpha_o = 1. \qquad [2.3]$$

The income multiples α_y, α_m, and α_o are parameterized to specify a hump-shaped life-cycle income profile described by a single parameter γ:

$$\alpha_y = 1 - \beta\gamma, \quad \alpha_m = 1 + (1 + \beta)\gamma, \quad \text{and} \quad \alpha_o = 1 - \gamma. \qquad [2.4]$$

The income multiples are all well-defined and strictly positive for any $0 < \gamma < 1$. The general pattern is depicted in Figure 1. As $\gamma \to 0$, the economy approaches a limiting case where all individuals alive at the same time receive the same income, and as $\gamma \to 1$, the difference in income over the life-cycle is at its maximum with old individuals receiving a zero income. Immediate values of γ imply life-cycles that lie between these extremes, thus the parameter γ can be interpreted as the gradient of individuals' life-cycle income profiles. The presence of the parameter β in [2.4] implies that the income gradient from young to middle-aged is less than the gradient from middle-aged to old.[1]

There is no government spending and no international trade, and the composite good is not storable, hence the goods-market clearing condition is

$$C_t = Y_t. \qquad [2.5]$$

Let $g_t \equiv (Y_t - Y_{t-1})/Y_{t-1}$ denote the growth rate of aggregate income between period $t - 1$ and t. It is assumed that the fluctuations in the stochastic process $\{g_t\}$ are bounded in the sense that in every time period and in every state of the world, no generation has a monotonic life-cycle income

[1] Introducing this feature implies that the steady state of the model will have some convenient properties. See Proposition 1 in section 3.

path. This requires that fluctuations in aggregate income are never large enough to dominate the variation in individual incomes over the life-cycle.[2]

The economy has a central bank that defines a reserve asset, referred to as 'money'. Reserves held between period t and $t+1$ are remunerated at a nominal interest rate i_t known in advance at time t. The economy is 'cash-less' in that money is not required for transactions, but money is used by agents as a unit of account in pricing and in financial contracts. One unit of goods costs P_t units of money at time t, and $\pi_t \equiv (P_t - P_{t-1})/P_{t-1}$ denotes the inflation rate between period $t-1$ and t. Monetary policy is specified as a rule for setting the nominal interest rate, for example a Taylor rule:

$$1 + i_t = \psi_0(1+\pi_t)^{\psi_\pi}, \tag{2.6}$$

where the coefficient ψ_π measures the sensitivity of nominal interest rates to inflation movements. It is assumed the Taylor principle $\psi_\pi > 1$ is satisfied. Finally, the central bank maintains a net supply of reserves equal to zero in equilibrium.

2.1 Incomplete markets

Asset markets are assumed to be incomplete. No individual can short-sell state-contingent bonds (Arrow-Debreu securities), and hence in equilibrium, no individual can buy such securities. The only asset that can be traded is a one-period, nominal, non-contingent bond. Individuals can take positive or negative positions in this bond (save or borrow), and there is no limit on borrowing other than being able to repay in all states of the world given non-negativity constraints on consumption. Hence no default occurs, and the bonds are therefore risk free in nominal terms. Bonds that have a nominal face value of 1 paying off at time $t+1$ trade at price Q_t in terms of money at time t. These bonds are perfect substitutes for the reserve asset defined by the central bank, so the absence of arbitrage opportunities requires that

$$Q_t = \frac{1}{1+i_t}. \tag{2.7}$$

The central bank's interest-rate policy thus sets the nominal price of the bonds.

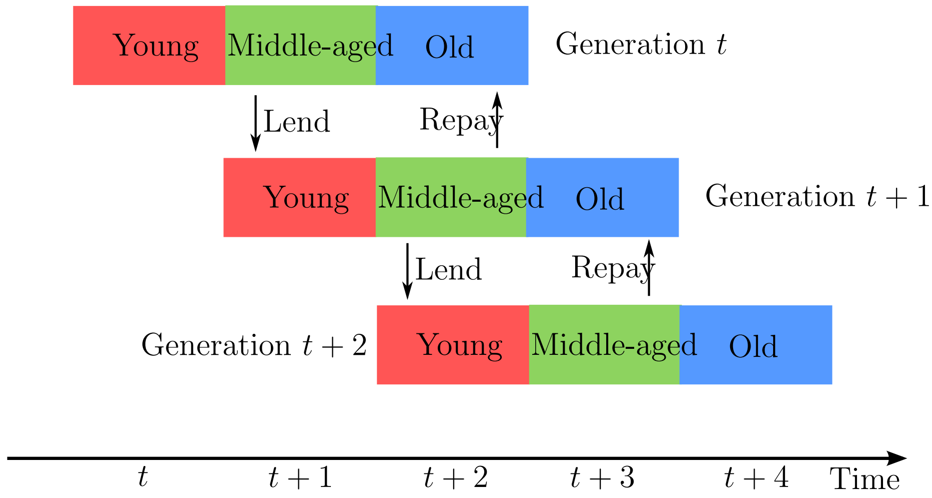
Let $B_{\mathrm{y},t}$ and $B_{\mathrm{m},t}$ denote the net bond positions per person of the young and middle-aged at the end of time t. The absence of intergenerational altruism implies that there will be no bequests ($B_{\mathrm{o},t} = 0$) and the young will begin life with no assets. The budget identities of the young, middle-aged, and old are respectively:

$$C_{\mathrm{y},t} + \frac{Q_t}{P_t}B_{\mathrm{y},t} = Y_{\mathrm{y},t}, \quad C_{\mathrm{m},t} + \frac{Q_t}{P_t}B_{\mathrm{m,t}} = Y_{\mathrm{m},t} + \frac{1}{P_t}B_{\mathrm{y},t-1}, \quad \text{and} \;\; C_{\mathrm{o},t} = Y_{\mathrm{o},t} + \frac{1}{P_t}B_{\mathrm{m},t-1}. \tag{2.8}$$

Maximizing the expected value $\mathbb{E}_t\mathscr{U}_t$ of the lifetime utility function [2.1] for each generation with

[2] The required bound on the fluctuations in g_t will be smaller when the the life-cycle gradient parameter γ is smaller.

Figure 2: *Saving and borrowing patterns*



respect to its bond holdings, subject to the budget identities [2.8], implies the Euler equations:

$$C_{\mathrm{y},t}^{-\frac{1}{\sigma}} = \beta \frac{1}{Q_t} \mathbb{E}_t \left[\frac{P_t}{P_{t+1}} C_{\mathrm{m},t+1}^{-\frac{1}{\sigma}} \right], \quad \text{and} \quad C_{\mathrm{m},t}^{-\frac{1}{\sigma}} = \beta \frac{1}{Q_t} \mathbb{E}_t \left[\frac{P_t}{P_{t+1}} C_{\mathrm{o},t+1}^{-\frac{1}{\sigma}} \right]. \qquad [2.9]$$

There is assumed to be no issuance of bonds by the government, so the bond market clearing condition is

$$\frac{1}{3} B_{\mathrm{y},t} + \frac{1}{3} B_{\mathrm{m},t} = 0. \qquad [2.10]$$

The equilibrium saving and borrowing patterns in the economy are depicted in Figure 2. Given the life-cycle income pattern in Figure 1, the young would like to borrow and the middle-aged would like to save, with the young repaying when they are middle-aged and the middle-aged receiving the repayment when they are old.[3]

It is convenient to introduce variables measured relative to GDP Y_t. These are denoted with lower-case letters. The consumption-to-GDP ratios for each generation are $c_{\mathrm{y},t}$, $c_{\mathrm{m},t}$, and $c_{\mathrm{o},t}$. It is also convenient to introduce one variable that measures the *gross* amount of bonds issued.[4] Let $B_t \equiv B_{\mathrm{m},t}$ denote the amount of bonds purchased by the middle-aged at time t, which will be equal in equilibrium to the amount issued by the young. Thus, the gross level of borrowing in the economy is B_t, with $D_t \equiv Q_t B_t / P_t$ being the real value of this debt at the time of issuance. The debt-to-GDP

[3]This type of exchange would not be feasible in an overlapping generations model with two-period lives. In that environment, saving is only possible by acquiring a physically storable asset or holding an 'outside' financial asset such as fiat money or government bonds. While the three-period lives OLG model of Samuelson (1958) also has the feature that saving requires an 'outside' asset, the life-cycle income profile there is monotonic. With a non-monotonic life-cycle income profile, trade between generations is possible even with only 'inside' financial assets. As will be seen, under the assumptions of the model here, there is no Pareto-improvement from introducing an 'outside' asset because while the equilibrium will be inefficient, it will not be *dynamically* inefficient.

[4]The net bond positions of the household sector and the whole economy are of course zero under the assumptions made.

ratio is denoted by d_t. These definitions are listed below for reference:

$$c_{\mathrm{y},t} \equiv C_{\mathrm{y},t}/Y_t, \quad c_{\mathrm{m},t} \equiv C_{\mathrm{m},t}/Y_t, \quad c_{\mathrm{o},t} \equiv C_{\mathrm{o},t}/Y_t, \quad \text{and} \quad d_t \equiv \frac{Q_t B_t}{P_t Y_t}. \tag{2.11}$$

The *ex-post* real return r_t on nominal bonds between period $t-1$ and t is the nominal interest rate i_{t-1} adjusted for inflation π_t:

$$1 + r_t = \frac{1 + i_{t-1}}{1 + \pi_t}. \tag{2.12}$$

The *ex-ante* real interest rate ρ_t between periods t and $t+1$ is defined as the conditional expectation of the ex-post return between those periods:

$$1 + \rho_t = \mathbb{E}_t\left[\frac{1 + i_t}{1 + \pi_{t+1}}\right], \tag{2.13}$$

or equivalently, $\rho_t = \mathbb{E}_t r_{t+1}$.[5]

Using the age-specific incomes [2.3], bond-market clearing [2.10] (implying $B_{\mathrm{y},t} = -B_t$), the no-arbitrage condition [2.7], and the definitions in [2.11] and [2.12], the budget identities in [2.8] for each generation can be written as:

$$c_{\mathrm{y},t} = \alpha_{\mathrm{y}} + d_t, \quad c_{\mathrm{m},t} + d_t = \alpha_{\mathrm{m}} - \left(\frac{1 + r_t}{1 + g_t}\right) d_{t-1}, \quad \text{and} \quad c_{\mathrm{o},t} = \alpha_{\mathrm{o}} + \left(\frac{1 + r_t}{1 + g_t}\right) d_{t-1}, \tag{2.14a}$$

where g_t is the growth rate of GDP Y_t. Similarly, the Euler equations in [2.9] become:

$$\beta\mathbb{E}_t\left[(1 + r_{t+1})(1 + g_{t+1})^{-\frac{1}{\sigma}}\left(\frac{c_{\mathrm{m},t+1}}{c_{\mathrm{y},t}}\right)^{-\frac{1}{\sigma}}\right] = \beta\mathbb{E}_t\left[(1 + r_{t+1})(1 + g_{t+1})^{-\frac{1}{\sigma}}\left(\frac{c_{\mathrm{o},t+1}}{c_{\mathrm{m},t}}\right)^{-\frac{1}{\sigma}}\right] = 1. \tag{2.14b}$$

Finally, goods-market clearing [2.5] with the definition of aggregate consumption [2.2] requires:

$$\frac{1}{3}c_{\mathrm{y},t} + \frac{1}{3}c_{\mathrm{m},t} + \frac{1}{3}c_{\mathrm{o},t} = 1. \tag{2.15}$$

Taking as given GDP growth g_t (with this variable being treated as exogenous for now), equations [2.14a]–[2.14b] together with the ex-post Fisher equation [2.12] and a specification of monetary policy, such as [2.6], the variables $c_{\mathrm{y},t}$, $c_{\mathrm{m},t}$, $c_{\mathrm{o},t}$, d_t, r_t, i_t, and π_t can in principle be determined. By Walras' law, the goods market clearing condition [2.15] is redundant, so this equation is dropped.

Before studying the equilibrium of the economy under different monetary policies, it is helpful to study the hypothetical world of complete financial markets as a benchmark.

2.2 The complete markets benchmark

Suppose it were possible for individuals to take short and long positions in a range of Arrow-Debreu securities for each possible state of the world. Without loss of generality, assume the payoffs of

[5] This real interest rate is important for saving and borrowing decisions, but there is no actual real risk-free asset to invest in.

these securities are specified in terms of real consumption, and their prices are quoted in real terms. Assume (again without loss of generality) the securities are traded sequentially for states of the world that will be realized one period in the future. Let K_{t+1} denote the kernel of prices for securities with payoffs of one unit of consumption at time $t+1$ in terms of consumption at time t. The prices are given relative to the (conditional) probabilities of each state.

Let $S_{\mathrm{y},t+1}$ and $S_{\mathrm{m},t+1}$ denote the per-person net positions in the Arrow-Debreu securities at the end of period t of the young and middle-aged respectively at time t (with $S_{\mathrm{o},t+1} = 0$ for the old, who hold no assets at the end of period t). These variables give the real payoffs individuals will receive (or make, if negative) at time $t+1$. The price of taking net position S_{t+1} at time t is $\mathbb{E}_t K_{t+1} S_{t+1}$ (this is the amount received from selling securities if negative).

In what follows, the the levels of consumption obtained with the complete markets with an asterisk to distinguish them from the consumption outcomes with incomplete markets. The budget constraints of the young, middle-aged, and old are:

$$C^*_{\mathrm{y},t} + \mathbb{E}_t K_{t+1} S_{\mathrm{y},t+1} = Y_{\mathrm{y},t}, \quad C^*_{\mathrm{m},t} + \mathbb{E}_t K_{t+1} S_{\mathrm{m},t+1} = Y_{\mathrm{m},t} + S_{\mathrm{y},t}, \quad \text{and} \;\; C^*_{\mathrm{o},t} = Y_{\mathrm{o},t} + S_{\mathrm{m},t}. \qquad [2.16]$$

Maximizing utility [2.1] for each generation with respect to holdings of Arrow-Debreu securities, subject to the budget constraints [2.16], implies the Euler equations:

$$\beta \left(\frac{C^*_{\mathrm{m},t+1}}{C^*_{\mathrm{y},t}} \right)^{-\frac{1}{\sigma}} = K_{t+1}, \quad \text{and} \;\; \beta \left(\frac{C^*_{\mathrm{o},t+1}}{C^*_{\mathrm{m},t}} \right)^{-\frac{1}{\sigma}} = K_{t+1}, \qquad [2.17]$$

where these hold for all states of the world at time $t+1$. Market clearing for the Arrow-Debreu securities requires:

$$\frac{1}{3} S_{\mathrm{y},t} + \frac{1}{3} S_{\mathrm{m},t} = 0. \qquad [2.18]$$

Let $c^*_{\mathrm{y},t}$, $c^*_{\mathrm{m},t}$, and $c^*_{\mathrm{o},t}$ denote the complete-markets generation-specific levels of consumption relative to aggregate output Y_t, analogous to those defined in [2.11] for the case of incomplete markets. Let $S_{t+1} \equiv S_{\mathrm{m},t+1}$ denote the gross quantities of Arrow-Debreu securities outstanding at the end of period t, and let $D^*_t = \mathbb{E}_t K_{t+1} S_{t+1}$ be the real value of these securities at current market prices. The value of the securities relative to GDP is denoted by d^*_t, and the ex-post real return on the portfolio by r^*_t:

$$d^*_t = \frac{D^*_t}{Y_t} = \frac{\mathbb{E}_t K_{t+1} S_{t+1}}{Y_t}, \quad \text{and} \;\; 1 + r^*_t = \frac{S_t}{D^*_{t-1}} = \frac{S_t}{\mathbb{E}_{t-1} K_t S_t}. \qquad [2.19]$$

Using the age-specific income levels from [2.3], market-clearing for Arrow-Debreu securities [2.18] (implying $S_{\mathrm{y},t+1} = -S_{t+1}$), and the definitions in [2.19], the budget constraints [2.16] can be expressed as follows:

$$c^*_{\mathrm{y},t} = \alpha_{\mathrm{y}} + d^*_t, \quad c^*_{\mathrm{m},t} + d^*_t = \alpha_{\mathrm{m}} - \left(\frac{1 + r^*_t}{1 + g_t} \right) d^*_{t-1}, \quad \text{and} \;\; c^*_{\mathrm{o},t} = \alpha_{\mathrm{o}} + \left(\frac{1 + r^*_t}{1 + g_t} \right) d^*_{t-1}, \qquad [2.20a]$$

Using the definitions in [2.19], the Euler equations [2.17] imply

$$\beta\mathbb{E}_t\left[(1+r^*_{t+1})(1+g_{t+1})^{-\frac{1}{\sigma}}\left(\frac{c^*_{\mathrm{m},t+1}}{c^*_{\mathrm{y},t}}\right)^{-\frac{1}{\sigma}}\right]=\beta\mathbb{E}_t\left[(1+r^*_{t+1})(1+g_{t+1})^{-\frac{1}{\sigma}}\left(\frac{c^*_{\mathrm{o},t+1}}{c^*_{\mathrm{m},t}}\right)^{-\frac{1}{\sigma}}\right]=1, \qquad [2.20b]$$

and

$$\frac{c^*_{\mathrm{m},t+1}}{c^*_{\mathrm{y},t}}=\frac{c^*_{\mathrm{o},t+1}}{c^*_{\mathrm{m},t}}. \qquad [2.20c]$$

Note that equations [2.20a]–[2.20b] have exactly the same form as their incomplete-markets counterparts [2.14a]–[2.14b]. The distinctive feature of complete markets is that equation [2.20c] also holds (note that with [2.20c], one of the equations in [2.20b] is redundant, so there are only two independent equations in [2.20b]–[2.20c]). There is also the equivalent of equation [2.15], which remains redundant by Walras' law.

As will be seen, the system of equations comprising [2.20a]–[2.20c] determines the variables $c^*_{\mathrm{m},t}$, $c^*_{\mathrm{o},t}$, d^*_t, r^*_t taking aggregate real growth g_t as given. Intuitively, since markets are complete (and real GDP is assumed exogenous), monetary policy cannot affect real consumption allocations or financial-market variables in real terms. The absence of arbitrage opportunities implies that a risk-free nominal bond would have price $Q_t = \mathbb{E}_t[K_{t+1}(P_t/P_{t+1})]$ in terms of money. Using [2.7] and [2.17], the nominal interest rate must satisfy the following asset-pricing equation:

$$\beta(1+i_t)\mathbb{E}_t\left[\frac{(1+g_{t+1})^{-\frac{1}{\sigma}}}{(1+\pi_{t+1})}\left(\frac{c^*_{\mathrm{m},t+1}}{c^*_{\mathrm{y},t}}\right)^{-\frac{1}{\sigma}}\right]=1. \qquad [2.21]$$

The equation above, together with a specification of monetary policy such as equation [2.6], determines the nominal interest rate i_t and inflation π_t.

2.3 Pareto-efficient allocations

Now consider the economy from the perspective of a social planner who has the power to mandate allocations of consumption to specific individuals (by making the appropriate transfers). The planner is utilitarian and maximizes a weighted sum of individual utilities subject to the economy's resource constraint.

Starting at some time t_0, the Pareto weight assigned to the generation born at time t is denoted by $\beta^{t-t_0}\omega_t/3$, where the term ω_t is scaled by the subjective discount factor β^{t-t_0} between time t_0 and t, and the population share $1/3$ of that generation at any point when its members are alive (the scaling is without loss of generality since ω_t has not been specified). The weight ω_t can be a function of the state of the world at time t when the corresponding generation is born, but does not depend on the state of the world to be realized at times $t+1$ and $t+2$. Thus, a generation's Pareto weight does not change during its lifetime (but is generally not known prior to its birth). This implies that the notion of efficiency here is *ex-ante* efficiency judged from the standpoint of each generation at birth (but not prior to birth).[6]

The social welfare function for a planner starting at time t_0 is:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0}\left[\frac{1}{3}\sum_{t=t_0-2}^{\infty} \beta^{t-t_0}\omega_t \mathscr{U}_t\right]. \qquad [2.22]$$

The Lagrangian for maximizing social welfare subject to the economy's resource constraint (given by [2.2] and [2.5]) is:

$$\mathcal{L}_{t_0} = \mathbb{E}_{t_0}\left[\frac{1}{3}\sum_{t=t_0-2}^{\infty} \beta^{t-t_0}\omega_t \mathscr{U}_t + \sum_{t=t_0}^{\infty}\beta^{t-t_0} Y_t^{-\frac{1}{\sigma}} \aleph_t \left(Y_t - \frac{1}{3}C_{\mathrm{y},t} - \frac{1}{3}C_{\mathrm{m},t} - \frac{1}{3}C_{\mathrm{o},t}\right)\right], \qquad [2.23]$$

where the Lagrangian multiplier on the time-t resource constraint is $\beta^{t-t_0}Y_t^{-\frac{1}{\sigma}}\aleph_t$ (where the scaling by $\beta^{t-t_0}Y_t^{-\frac{1}{\sigma}}$ is for convenience). Using the utility function [2.1], the first-order conditions with respect to age-specific consumption levels $C_{\mathrm{y},t}$, $C_{\mathrm{m},t}$ and $C_{\mathrm{o},t}$ are:

$$\omega_t C_{\mathrm{y},t}^{-\frac{1}{\sigma}} = Y_t^{-\frac{1}{\sigma}}\aleph_t, \quad \omega_{t-1} C_{\mathrm{m},t}^{-\frac{1}{\sigma}} = Y_t^{-\frac{1}{\sigma}}\aleph_t, \quad \text{and} \;\; \omega_{t-2} C_{\mathrm{o},t}^{-\frac{1}{\sigma}} = Y_t^{-\frac{1}{\sigma}}\aleph_t \;\; \text{for all} \;\; t \geq t_0. \qquad [2.24]$$

Manipulating these first-order conditions and using the definitions of age-specific consumption relative to aggregate income from [2.11]:

$$\left(\frac{c_{\mathrm{m},t+1}}{c_{\mathrm{y},t}}\right)^{-\frac{1}{\sigma}} = \left(\frac{c_{\mathrm{o},t+1}}{c_{\mathrm{m},t}}\right)^{-\frac{1}{\sigma}} = \frac{\aleph_{t+1}}{\aleph_t} \;\; \text{for all} \;\; t \geq t_0, \qquad [2.25]$$

where these equations hold in all states of the world at date $t+1$.

Now consider the complete-markets case from section 2.2. Since equation [2.20c] holds, it can be seen using [2.24] that the implied consumption allocation is ex-ante Pareto efficient, being supported by the following Pareto weights ω_t^*:

$$\omega_t^* = \frac{\aleph_t^*}{c_{\mathrm{y},t}^{*\,-\frac{1}{\sigma}}}, \quad \text{where} \;\; \aleph_t^* = \mathbb{E}_t\left[\prod_{\ell=1}^{\infty}\left(\beta(1+r_{t+\ell}^*)(1+g_{t+\ell})^{-\frac{1}{\sigma}}\right)\right], \qquad [2.26]$$

with the equation for $\aleph_t^*$ derived from [2.20b] and [2.25].

To what extent can monetary policy achieve Pareto efficiency in a world of incomplete markets? Could monetary policy be used to achieve the complete-markets equilibrium characterized by equations [2.20a]–[2.20c]? Given the form of the incomplete-markets equilibrium conditions [2.14a]–[2.14b], all that monetary policy needs to do is ensure that equation [2.20c] holds. With one equation to satisfy and one instrument, this should be possible. Intuitively, monetary policy will use inflation to manipulate the debt-GDP ratio d_t to ensure [2.20c] holds in equilibrium. Thus, in principle, monetary policy can achieve one ex-ante Pareto-efficient allocation. Are other Pareto-efficient allo-

[6] This restriction is needed to make the notion of efficiency relevant. Otherwise with a fixed amount of goods each period to allocate, *any* non-wasteful allocation would always be *ex-post* Pareto efficient.

cations attainable? Since ex-ante efficiency requires equation [2.25] must hold, and since this implies [2.20c], it follows that the *only* efficient allocation that might be implemented through monetary policy is the complete-markets equilibrium.

3 Endowment economy

To understand the analysis of the model, it is helpful first to characterize the equilibrium and the effects of monetary policy in an endowment economy. The general case of a production economy is considered in a later section. Thus, $Y_t = A_t$, where A_t is an exogenous stochastic process for the endowment.

The equilibrium will also be approximated for fluctuations in the growth rate g_t of aggregate output around a zero-growth rate steady state ($\bar{g} = 0$).[7] It is assumed that aggregate real income growth is bounded with probability one, that is, $|g_t| \leq \Gamma$ for some bound Γ. This rules out non-stationarity in income growth, but the level of income can be either stationary or non-stationary depending on the specification of the stochastic process for A_t ($g_t = (A_t - A_{t-1})/A_t$).

There are no idiosyncratic shocks, though aggregate shocks have different effects on different generations, which in general they are not able completely to insure themselves against.

3.1 The steady state

The first step is to characterize when a unique steady state exists, and what are its characteristics if so.

Proposition 1 *The following hold whether or not markets are complete:*

(i) There exists a steady state where consumption is equal to per-person aggregate income for all generations:

$$\bar{c}_{\mathrm{y}} = \bar{c}^*_{\mathrm{y}} = 1, \quad \bar{c}_{\mathrm{m}} = \bar{c}^*_{\mathrm{m}} = 1, \quad \textit{and} \;\; \bar{c}_{\mathrm{o}} = \bar{c}^*_{\mathrm{o}} = 1,$$

and in which the debt-to-GDP ratio is positive, and the real return on bonds is equal to individuals' rate of time preference:

$$\bar{d} = \bar{d}^* = \beta\gamma, \quad \textit{and} \;\; \bar{r} = \bar{r}^* = \varrho.$$

Inflation is given by:

$$1 + \bar{\pi} = \left(\frac{1+\varrho}{\psi_0}\right)^{\frac{1}{\psi_\pi - 1}}.$$

(ii) A sufficient condition for uniqueness of the steady state above is $\sigma \geq 1/2$, and a necessary condition is $\sigma > \left(\frac{1}{2+\varrho}\right)\gamma$.

[7] There is no reason in principle why the aggregate real growth rate needs to be zero, but this is assumed for simplicity. For reasonable assumptions on average growth, this is not likely to be important quantitatively.

PROOF See appendix A.1. ■

It is assumed throughout that the parameters used are such that there is a unique steady state.

3.2 Fluctuations around the steady state

Log deviations of variables from their steady-state values are denoted with sans serif letters, for example, $\mathsf{d}_t \equiv \log d_t - \log \bar{d}$. For all variables that are either interest rates or growth rates, the log deviation is of the gross rate, for example, $\mathsf{g}_t \equiv \log(1+g_t)-\log(1+\bar{g})$ and $\mathsf{r}_t \equiv \log(1+r_t)-\log(1+\bar{r})$. For all variables that do not necessarily have a steady state, the sans serif equivalent denotes simply the logarithm of the variable, for example, $\mathsf{Y}_t \equiv \log Y_t$.

With these definitions and the nature of the steady state characterized in Proposition 1, aggregate income growth is $\mathsf{g}_t = \mathsf{Y}_t - \mathsf{Y}_{t-1}$, where $\mathsf{Y}_t = \mathsf{A}_t$ in terms of the exogenous endowment A_t. The growth rate satisfies the bound $|\mathsf{g}_t| \leq \log(1+\Gamma)$ with probability one, in terms of the bound Γ. The variables defined as ratios to GDP in [2.11] are such that $\mathsf{c}_{\mathrm{y},t} = \mathsf{C}_{\mathrm{y},t} - \mathsf{Y}_t$, $\mathsf{c}_{\mathrm{m},t} = \mathsf{C}_{\mathrm{m},t} - \mathsf{Y}_t$, $\mathsf{c}_{\mathrm{o},t} = \mathsf{C}_{\mathrm{o},t} - \mathsf{Y}_t$, and $\mathsf{d}_t = \mathsf{D}_t - \mathsf{Y}_t$.

In the following, equations are log-linearized around the steady state and terms that are second-order or higher in deviations from the steady state are suppressed. The Fisher identity [2.12] for the ex-post real return rate on nominal bonds becomes

$$\mathsf{r}_t = \mathsf{i}_{t-1} - \pi_t, \tag{3.1}$$

where $\pi_t = \mathsf{P}_t - \mathsf{P}_{t-1}$ is inflation. The ex-ante real interest rate from [2.13] is such that $\rho_t = \mathbb{E}_t \mathsf{r}_{t+1}$.

Making use of the parameterization [2.4] of the lifecycle income process with gradient γ, the budget constraints [2.14a] for each generation take the following log-linear expressions:

$$\mathsf{c}_{\mathrm{y},t} = \beta\gamma\mathsf{d}_t, \quad \mathsf{c}_{\mathrm{m},t} = -\beta\gamma\mathsf{d}_t - \gamma(\mathsf{r}_t - \mathsf{g}_t + \mathsf{d}_{t-1}), \quad \text{and} \quad \mathsf{c}_{\mathrm{o},t} = \gamma(\mathsf{r}_t - \mathsf{g}_t + \mathsf{d}_{t-1}). \tag{3.2a}$$

The log-linearized Euler equations [2.14b] for each generation are:

$$\mathsf{c}_{\mathrm{y},t} = \mathbb{E}_t \mathsf{c}_{\mathrm{m},t+1} - \sigma\mathbb{E}_t \mathsf{r}_{t+1} + \mathbb{E}_t \mathsf{g}_{t+1}, \quad \text{and} \quad \mathsf{c}_{\mathrm{m},t} = \mathbb{E}_t \mathsf{c}_{\mathrm{o},t+1} - \sigma\mathbb{E}_t \mathsf{r}_{t+1} + \mathbb{E}_t \mathsf{g}_{t+1}. \tag{3.2b}$$

The log-linearized goods market clearing condition [2.15] is the requirement $\mathsf{c}_{\mathrm{y},t}+\mathsf{c}_{\mathrm{m},t}+\mathsf{c}_{\mathrm{o},t} = 0$, which is seen to be automatically satisfied given the budget constraints in [3.2a]. Since each individual's consumption must also satisfy a non-negativity constraint, none of the log deviations $\mathsf{c}_{\mathrm{y},t}$, $\mathsf{c}_{\mathrm{m},t}$, or $\mathsf{c}_{\mathrm{o},t}$ can grow without bound.

The system of five equations in [3.2a]–[3.2b] contains five endogenous variables $\mathsf{c}_{\mathrm{y},t}$, $\mathsf{c}_{\mathrm{m},t}$, $\mathsf{c}_{\mathrm{o},t}$, r_t, and d_t, and one exogenous variable g_t. These equations can be solved subject to the requirement that all of $\mathsf{c}_{\mathrm{y},t}$, $\mathsf{c}_{\mathrm{m},t}$, and $\mathsf{c}_{\mathrm{o},t}$ are bounded.

Proposition 2 *Suppose that the parameters β, γ, and σ are such that the steady state described in Proposition 1 is unique. The class of solutions to equations [3.2a]–[3.2b] with the restriction that the variables $\mathsf{c}_{\mathrm{y},t}$, $\mathsf{c}_{\mathrm{m},t}$, and $\mathsf{c}_{\mathrm{o},t}$ must be bounded is given by:*

(i) The debt-GDP ratio evolves over time according to

$$\mathsf{d}_t = \lambda \mathsf{d}_{t-1} - \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(2\mathbb{E}_{t-1}\mathsf{f}_t + \mathsf{f}_{t-1}) + \nu_t, \qquad [3.3]$$

where ν_t is any bounded martingale difference stochastic process ($\mathbb{E}_{t-1}\nu_t = 0$) and f_t is the stochastic process defined by

$$\mathsf{f}_t = \sum_{\ell=1}^{\infty} \zeta^{\ell-1}\mathbb{E}_t \mathsf{g}_{t+\ell}, \qquad [3.4]$$

and with coefficients δ, λ, and ζ that depend on parameters β, γ, and σ as follows:

$$\delta = \frac{1}{2}\left((1+2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right); \qquad [3.5a]$$

$$\lambda = \frac{2\left(\frac{\beta\gamma}{\sigma} - 1\right)}{(1+2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}; \qquad [3.5b]$$

$$\zeta = \frac{2\beta\left(\frac{\gamma}{\sigma} - 1\right)}{(1+2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}. \qquad [3.5c]$$

The coefficients are such that $0 < \delta \leq 2(1+\beta)$, $|\lambda| < \beta$, and $|\zeta| < 1$.

(ii) Given a solution for the debt-GDP ratio d_t from [3.3], the ex-post real return r_t must satisfy the equation:

$$\mathsf{r}_t = -\theta \mathsf{d}_t - \mathsf{d}_{t-1} + \mathsf{g}_t - \vartheta\left(\frac{1-\sigma}{\sigma}\right)\mathsf{f}_t, \qquad [3.6]$$

where f_t is as defined in [3.4] and the coefficients θ and ϑ depend on the parameters β, γ, and σ as follows:

$$\theta = \frac{(1+2\beta) + \left(1 + \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}{2\left(1 + \frac{\gamma}{\sigma}\right)}; \qquad [3.7a]$$

$$\vartheta = \frac{2\left((2+\beta) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right)}{\left(1 + \frac{\gamma}{\sigma}\right)\left((1+2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right)}. \qquad [3.7b]$$

The coefficients satisfy the restrictions $\beta \leq \theta \leq \delta$ and $\vartheta > 0$.

PROOF See appendix A.3 ■

In spite of having the same number of equations as endogenous variables, the presence of expectations of the future in the Euler equations [3.2b] combined with more stable eigenvalues than predetermined variables (only d_{t-1} in the budget constraints [3.2a]) means that the evolution of the debt-GDP ratio d_t is determined only up to a martingale difference ν_t. This means that [3.3] uniquely determines the conditional expectation of the future debt-GDP ratio:

$$\mathbb{E}_{t-1}d_t = \lambda d_{t-1} - \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(2\mathbb{E}_{t-1}f_t + f_{t-1}). \quad [3.8]$$

For a particular innovation ν_t, the ex-post real return r_t is determined by [3.6], and the age-specific consumption-GDP ratios are determined by the budget constraints [3.2a].

The martingale difference ν_t can itself only be determined by augmenting the system of equations with the Fisher equation [3.1] and a description of monetary policy such as the Taylor rule [2.6], which has the following log-linear form:

$$i_t = \psi_\pi \pi_t. \quad [3.9]$$

Note that there are two limiting cases in which the economy (up to a log-linear approximation) reduces to a representative-agent economy, formally, where $C_{y,t} = C_{m,t} = C_{o,t} = C_t = Y_t$, and where the representative-agent consumption Euler equation $\mathbb{E}_t g_{t+1} = -\sigma\mathbb{E}_t r_{t+1}$, that is:

$$Y_t = \mathbb{E}_t Y_{t+1} - \sigma(i_t - \mathbb{E}_t\pi_{t+1}).$$

These special cases are when the gradient of the lifecycle income profile tends to zero ($\gamma \to 0$) so that individuals of all ages receive the same income, and when the intertemporal elasticity of substitution tends to infinity ($\sigma \to \infty$, linear utility). This conclusion follows from inspection of equations [3.2a], [3.2b], [3.3], and [3.6].

3.3 Equilibrium with complete markets

Before taking up the question of how monetary policy affects the economy's debt dynamics under incomplete markets, it is useful to characterize the equilibrium with complete markets, which represent one ex-ante Pareto efficient allocation. With complete markets, the system of equations is [2.20a]–[2.20c]. Equations [2.20a]–[2.20b] have exactly the same form as their incomplete-markets equivalents [2.14a]–[2.14b], so their log-linear approximations are as in [3.2a]–[3.2b] with $c_{y,t}$, $c_{m,t}$, $c_{o,t}$, r_t, and d_t replaced by $c^*_{y,t}$, $c^*_{m,t}$, $c^*_{o,t}$, r^*_t, and d^*_t, where the asterisk is used to distinguish the complete-markets outcome:

$$c^*_{y,t} = \beta\gamma d^*_t, \quad c^*_{m,t} = -\beta\gamma d^*_t - \gamma(r^*_t - g_t + d^*_{t-1}), \quad c^*_{o,t} = \gamma(r^*_t - g_t + d^*_{t-1}); \quad [3.10a]$$

$$c^*_{y,t} = \mathbb{E}_t c^*_{m,t+1} - \sigma\mathbb{E}_t r^*_{t+1} + \mathbb{E}_t g_{t+1}, \quad \text{and} \quad c^*_{m,t} = \mathbb{E}_t c^*_{o,t+1} - \sigma\mathbb{E}_t r^*_{t+1} + \mathbb{E}_t g_{t+1}. \quad [3.10b]$$

In addition, with complete markets, equation [2.20c] must hold in all states of the world, which has the following log-linear form:

$$c^*_{m,t+1} - c^*_{y,t} = c^*_{o,t+1} - c^*_{m,t}. \quad [3.10c]$$

It can be seen from [3.2b] that this equation holds in expectation under incomplete markets. Under complete markets, it also holds in realization. This allows the free martingale difference ν_t in the incomplete-markets debt dynamics to be determined.

The following result describes the equilibrium with complete markets and characterizes the 'gaps' between the incomplete and complete markets outcomes. These 'gaps' are denoted with a tilde, for example, $\tilde{d}_t \equiv d_t - d_t^*$ and $\tilde{r}_t \equiv r_t - r_t^*$.

Proposition 3 *The solution to the system of equations [3.2a], [3.2b], and [3.10c] is as follows.*

(i) The debt-GDP ratio d_t^* evolves over time according to

$$d_t^* = \lambda d_{t-1}^* - \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(2f_t + f_{t-1}), \quad [3.11]$$

where f_t is as defined in [3.4], and the coefficients δ and θ as given in [3.5a]–[3.7a]. The gap between the debt-GDP ratios with incomplete and complete markets must satisfy the equation:

$$\mathbb{E}_t \tilde{d}_{t+1} = \lambda \tilde{d}_t. \quad [3.12]$$

(ii) The real return on the complete markets portfolio is:

$$r_t^* = \lambda r_{t-1}^* + (g_t - \lambda g_{t-1}) + \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(\beta f_t + 2(1+\beta)f_{t-1} + f_{t-2}). \quad [3.13]$$

The relationship between the gaps for the real return and the debt-GDP ratio is:

$$\tilde{r}_t = -\theta \tilde{d}_t - \tilde{d}_{t-1}, \quad [3.14]$$

where the coefficient θ is given in [3.7a].

(iii) The gaps between the consumption allocations under incomplete markets are:

$$\tilde{C}_{y,t} = \beta\gamma\tilde{d}_t, \quad \tilde{C}_{m,t} = \gamma(\theta - \beta)\tilde{d}_t, \quad \text{and} \quad \tilde{C}_{o,t} = -\gamma\theta\tilde{d}_t. \quad [3.15]$$

(iv) In the case that either the intertemporal elasticity of substitution is one ($\sigma = 1$, log utility), or the conditional expectation of future real output growth is equal to its unconditional mean ($\mathbb{E}_t g_{t+1} = 0$), the complete-markets debt-GDP ratio is constant ($d_t^* = 0$) and the real return equal to the real growth rate ($r_t^* = g_t$).

Proof See appendix A.4 ■

The complete markets equilibrium is unsurprisingly invariant to monetary policy (the level of aggregate output is exogenous and the distribution of wealth is unaffected by monetary policy when individuals can hold securities with known real returns). As has been discussed, in a world of incomplete markets, the hypothetical complete-markets equilibrium is only ex-ante Pareto-efficient allocation that might be implemented through monetary policy. The gap between the consumption allocation under incomplete and complete markets is seen in Proposition 3 to be a function of the gap between the debt-GDP ratios under incomplete and complete markets. Implementation of the complete-markets consumption allocation is thus seen to be equivalent to closing this 'debt gap'.

3.4 Monetary policy

The impact of monetary policy on the debt gap can be studied with a system of three equations in three endogenous variables: inflation π_t, the nominal interest rate i_t, and the 'debt gap' $\tilde{\mathsf{d}}_t$. The first equation describes the dynamics of the debt gap. This is [3.12]. The second follows from the equilibrium ex-post real return using [3.1] and [3.14]:

$$\lambda\tilde{\mathsf{d}}_t = \mathbb{E}_t\tilde{\mathsf{d}}_{t+1}; \tag{3.16a}$$

$$\pi_t = \mathsf{i}_{t-1} + \theta\tilde{\mathsf{d}}_t + \tilde{\mathsf{d}}_{t-1} - \mathsf{r}^*_t. \tag{3.16b}$$

The third equation is a description of monetary policy, whether a Taylor rule of the form [3.9], or an alternative specification. The hypothetical real return r^*_t under complete markets depends only on the exogenous rate of GDP growth g_t according to [3.13].

To begin with, consider a monetary policy of strict inflation targeting. This is represented by the targeting rule $\pi_t = 0$, which means that the nominal interest rate i_t is adjusted so that this criterion holds at all times and in all states of the world. The effects of this policy are described below.

Proposition 4 *The unique equilibrium of the economy described by equations* [3.16a]–[3.16b] *together with strict inflation targeting rule* $\pi_t = 0$ *is:*

$$\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} + \frac{1}{\theta}\left(\mathsf{r}^*_t - \mathbb{E}_{t-1}\mathsf{r}^*_t\right), \tag{3.17a}$$

$$\mathsf{i}_t = \mathbb{E}_t\mathsf{r}^*_{t+1} - (1+\lambda\theta)\tilde{\mathsf{d}}_t, \tag{3.17b}$$

and constant inflation $\pi_t = 0$ *according to the target.*

Proof With constant inflation, equation [3.16b] implies

$$\theta(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t) - (\mathsf{r}^*_t - \mathbb{E}_{t-1}\mathsf{r}^*_t) = 0.$$

The equation [3.16a] for debt dynamics implies $\mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1}$. Putting these together yields [3.17a]. Taking the conditional expectation of equation [3.16b] dated $t+1$ with $\pi_{t+1} = 0$ yields:

$$\mathsf{i}_t + \theta\mathbb{E}_t\tilde{\mathsf{d}}_{t+1} + \tilde{\mathsf{d}}_t - \mathbb{E}_t\mathsf{r}^*_{t+1} = 0.$$

Substituting the expression for $\mathbb{E}_t\tilde{\mathsf{d}}_{t+1}$ from [3.16a] and rearranging leads to [3.17b]. This is the nominal interest rate that implements the strict inflation target. ■

This policy fails to close the 'debt gap' whenever the complete markets return r_t^* is not perfectly predictable. Using equation [3.13], the forecast error is:

$$r_t^* - \mathbb{E}_{t-1}r_t^* = (g_t - \mathbb{E}_{t-1}g_t) + \frac{\beta}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(f_t - \mathbb{E}_{t-1}f_t).$$

3.5 Optimal monetary policy

An obvious alternative policy is to adopt $\tilde{d}_t = 0$ (that is, $d_t = d_t^*$) as a targeting rule. Again, this means adjusting the nominal interest rate to ensure that this criterion holds. The following proposition characterizes the set of equilibria according to this targeting rule.

Proposition 5 *The set of equilibria of the economy described by equations [3.16a]–[3.16b] and the targeting rule $\tilde{d}_t = 0$ is:*

$$\pi_t = z_{t-1} - (r_t^* - \mathbb{E}_{t-1}r_t^*), \quad [3.18a]$$

$$i_t = z_t + \mathbb{E}_t r_{t+1}^*, \quad [3.18b]$$

with the target $\tilde{d}_t = 0$ met, and where z_t is any arbitrary stochastic process.

PROOF First, note that the target is always consistent with equation [3.16a]. Next, to achieve the target, equation [3.16b] implies that the unexpected component of inflation must satisfy

$$(\pi_t - \mathbb{E}_{t-1}\pi_t) = -(r_t^* - \mathbb{E}_{t-1}r_t^*).$$

Since monetary policy can always generate an arbitrary inflation process, this shows that the target is feasible. Now let $z_t \equiv \mathbb{E}_t\pi_{t+1}$ denote expectations of inflation one period ahead. Since $\pi_t = \mathbb{E}_{t-1}\pi_t + (\pi_t - \mathbb{E}_{t-1}\pi_t)$ by definition, equation [3.18a] follows. Taking conditional expectations of [3.16b] dated $t+1$ implies $\mathbb{E}_t\pi_{t+1} = i_t - \mathbb{E}_t r_{t+1}^*$. Rearranging and using the definition of z_t yields equation [3.18b]. This solution satisfies all the necessary equations, but places no restrictions on the stochastic process z_t for inflation expectations. ■

While this targeting rule achieves the optimal monetary policy $\tilde{d}_t = 0$, it fails to pin down inflation expectations. The problem is a familiar one: the policy focuses all attention on a real variable, the debt-GDP ratio, and thus provides no nominal anchor.

What would be desirable is a policy that achieves Pareto efficiency while also providing a nominal anchor. It turns out that nominal income targeting provides a solution to this challenge. Let $N_t \equiv P_t Y_t$ denote aggregate nominal GDP, or in log terms, $N_t \equiv P_t + Y_t$. Consider the following policy of targeting an exogenous path for nominal income.

Proposition 6 *The unique equilibrium of an economy described by equations [3.16a]–[3.16b] together with the targeting rule*

$$N_t = \frac{\beta}{2}d_t^*, \quad [3.19]$$

where d_t^* *is given in* [3.11], *is the following:*

$$\mathsf{P}_t = \frac{\beta}{2}\mathsf{d}_t^* - \mathsf{Y}_t, \qquad [3.20a]$$

$$\mathsf{i}_t = \mathbb{E}_t\mathsf{r}_{t+1}^* - \mathbb{E}_t\mathsf{g}_{t+1} + \frac{\beta}{2}\left(\mathbb{E}_t\mathsf{d}_{t+1}^* - \mathsf{d}_t^*\right), \qquad [3.20b]$$

and the debt gap is always closed ($\tilde{\mathsf{d}}_t = 0$).

PROOF The nominal income target [3.19] directly implies equation [3.20a] for the price level. Taking the first difference yields $\pi_t = (\beta/2)(\mathsf{d}_t^* - \mathsf{d}_{t-1}^*) - \mathsf{g}_t$, and thus the unexpected component of inflation is:

$$\pi_t - \mathbb{E}_{t-1}\pi_t = \frac{\beta}{2}\left(\mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^*\right) - (\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t). \qquad [3.21]$$

With $\mathsf{r}_t = \mathsf{i}_{t-1} + \pi_t$ it must be the case that $\mathsf{r}_t - \mathbb{E}_{t-1}\mathsf{r}_t = -(\pi_t - \mathbb{E}_{t-1}\pi_t)$, and thus that the unexpected change in the real return gap $\tilde{\mathsf{r}}_t$ is:

$$\tilde{\mathsf{r}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{r}}_t = -(\pi_t - \mathbb{E}_{t-1}\pi_t) - (\mathsf{r}_t^* - \mathbb{E}_{t-1}\mathsf{r}_t^*).$$

Substituting from [3.21] yields:

$$\tilde{\mathsf{r}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{r}}_t = (\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t) - \frac{\beta}{2}\left(\mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^*\right) - (\mathsf{r}_t^* - \mathbb{E}_{t-1}\mathsf{r}_t^*). \qquad [3.22]$$

Now observe from equations [3.11] and [3.13]:

$$\mathsf{d}_t^* - \mathbb{E}_{t-1}\mathsf{d}_t^* = -\frac{2}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(\mathsf{f}_t - \mathbb{E}_{t-1}\mathsf{f}_t), \quad \mathsf{r}_t^* - \mathbb{E}_{t-1}\mathsf{r}_t^* = (\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t) + \frac{\beta}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(\mathsf{f}_t - \mathbb{E}_{t-1}\mathsf{f}_t),$$

and hence $\tilde{\mathsf{r}}_t = \mathbb{E}_{t-1}\tilde{\mathsf{r}}_t$ by substituting these into [3.22]. Equation [3.14] implies $\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t = -(1/\theta)(\tilde{\mathsf{r}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{r}}_t)$, so together with [3.12] it follows that $\tilde{\mathsf{d}}_t = 0$ for all t. With $\tilde{\mathsf{d}}_t = 0$, equation [3.14] implies $\tilde{\mathsf{r}}_{t+1} = 0$, so $\mathsf{i}_t = \mathbb{E}_t\pi_{t+1} + \mathbb{E}_t\mathsf{r}_{t+1}^*$. The expression for the price level in [3.20a] implies $\mathbb{E}_t\pi_{t+1} = (\beta/2)(\mathbb{E}_t\mathsf{d}_{t+1}^* - \mathsf{d}_t^*) - \mathbb{E}_t\mathsf{g}_{t+1}$, which confirms [3.20b]. ■

3.6 Effects of monetary policy shocks

To understand the consequences of different monetary policies it is helpful to analyse the consequences of an exogenous change in the stance of monetary policy.

Proposition 7 *Suppose there are no fundamental shocks to real GDP growth* ($\mathsf{g}_t = 0$)*, but that there are exogenous shocks to the growth rate of nominal income targeted by monetary policy:*

$$\Delta\mathsf{N}_t = \epsilon_t, \quad \textit{where} \;\; \epsilon \sim i.i.d.(0, \varsigma^2). \qquad [3.23]$$

The equilibrium of the economy is:

$$\mathsf{r}_t = \mathsf{i}_{t-1} - \epsilon_t, \quad [3.24a]$$

$$\mathsf{i}_t = \rho_t = -(1+\lambda\theta)\tilde{\mathsf{d}}_t, \quad [3.24b]$$

$$\tilde{\mathsf{d}}_t = \lambda\tilde{\mathsf{d}}_{t-1} + \frac{1}{\theta}\epsilon_t. \quad [3.24c]$$

PROOF Nominal income is $\mathsf{N}_t = \mathsf{P}_t + \mathsf{Y}_t$. With no change in exogenous real GDP growth, $\Delta\mathsf{N}_t = \pi_t$, so it follows immediately that $\pi_t = \epsilon_t$. Equation [3.24a] then implied immediately by [3.1].

As $\mathsf{g}_t = 0$, it follows from [3.13] that $\mathsf{r}^*_t = 0$. Equating the unexpected components of both sides of [3.16b] then implies $\pi_t - \mathbb{E}_{t-1}\pi_t = \theta(\tilde{\mathsf{d}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{d}}_t)$. Since $\pi_t - \mathbb{E}_{t-1}\pi_t = \epsilon_t$, this leads to equation [3.24c].

Noting that $\mathbb{E}_t\pi_{t+1} = \mathbb{E}_t\epsilon_{t+1} = 0$ and taking expectations of [3.16b] at time $t+1$ implies $\mathsf{i}_t + \theta\mathbb{E}_t\tilde{\mathsf{d}}_{t+1} + \tilde{\mathsf{d}}_t = 0$. Then using [3.16a] confirms equation [3.24b]. Finally, with $\mathbb{E}_t\pi_{t+1} = 0$, [2.13] implies $\mathsf{i}_t = \rho_t$. ■

3.7 Inflation-indexed bonds

An alternative incomplete markets economy is one with only inflation-indexed bonds, that is, risk-free real bonds. Let $\rho^\dagger_t$ denote the real interest rate on such a bond between t and $t+1$. Repeating the earlier steps for risk-free nominal bonds, the generational budget constraints and Euler equations can be derived, where now the ex-post real return is simply the lagged real interest rate ($r^\dagger_t = \rho^\dagger_{t-1}$). These have the following log-linear expressions, where the notation † is used to distinguish the outcomes with indexed bonds:

$$\mathsf{c}^\dagger_{\mathrm{y},t} = \beta\gamma\mathsf{d}^\dagger_t, \quad \mathsf{c}^\dagger_{\mathrm{m},t} = -\beta\gamma\mathsf{d}^\dagger_t - \gamma(\rho^\dagger_{t-1} - \mathsf{g}_t + \mathsf{d}^\dagger_{t-1}), \quad \text{and} \quad \mathsf{c}^\dagger_{\mathrm{o},t} = \gamma(\rho^\dagger_{t-1} - \mathsf{g}_t + \mathsf{d}^\dagger_{t-1}), \quad [3.25a]$$

$$\mathsf{c}^\dagger_{\mathrm{y},t} = \mathbb{E}_t\mathsf{c}^\dagger_{\mathrm{m},t+1} - \sigma\rho^\dagger_t + \mathbb{E}_t\mathsf{g}_{t+1}, \quad \text{and} \quad \mathsf{c}^\dagger_{\mathrm{m},t} = \mathbb{E}_t\mathsf{c}^\dagger_{\mathrm{o},t+1} - \sigma\rho^\dagger_t + \mathbb{E}_t\mathsf{g}_{t+1}. \quad [3.25b]$$

Since these equations have the same form as [3.2a]–[3.2b] with $\mathsf{r}^\dagger_t = \rho^\dagger_{t-1}$, Proposition 2 can be applied and equation [3.3] must hold. But unlike the case of nominal bonds, the martingale difference ν_t can be determined without reference to monetary policy.

Proposition 8 *With only risk-free real bonds, the dynamics of the debt-GDP ratio gap relative to complete markets and the equilibrium real interest rate are:*

$$\tilde{\mathsf{d}}^\dagger_t = \lambda\tilde{\mathsf{d}}^\dagger_{t-1} + \frac{1}{\theta}\left(\mathsf{r}^*_t - \mathbb{E}_{t-1}\mathsf{r}^*_t\right), \quad [3.26a]$$

$$\rho^\dagger_t = \mathbb{E}_t\mathsf{r}^*_{t+1} - (1+\lambda\theta)\tilde{\mathsf{d}}^\dagger_t. \quad [3.26b]$$

The equilibrium is invariant to monetary policy.

Proof With $\mathsf{r}_t = \rho_{t-1}$, it follows that $\tilde{\mathsf{r}}_t - \mathbb{E}_{t-1}\tilde{\mathsf{r}}_t = -(\mathsf{r}_t^* - \mathbb{E}_{t-1}\mathsf{r}_t^*)$. Equation [3.26a] then follows from equating the unexpected components of both sides of [3.14] and using [3.12]. Equation [3.14] implies

$$\rho_t = \mathsf{r}_{t+1}^* - \theta\tilde{\mathsf{d}}_{t+1} - \mathsf{d}_t.$$

Taking conditional expectations of both sides and using [3.12] implies [3.26b]. ■

4 Sticky prices

The optimal monetary policy of nominal income targeting found in section 3 entails fluctuations in inflation. With fully flexible prices in product markets, this is without cost, but the conventional argument for inflation targeting is that such inflation fluctuations lead to misallocation of resources in goods and factor markets. This section adds sticky prices to the model to analyse optimal monetary policy subject to both incomplete financial markets and nominal rigidities in goods markets. To do this, it is necessary to introduce differentiated goods, imperfect competition, and a market for labour that can be hired by different firms.

4.1 Differentiated goods

Consumption in individuals' lifetime utility function [2.1] now denotes consumption of a composite good made up of a measure-one continuum of differentiated goods. Young, middle-aged, and old individuals share the same CES (Dixit-Stiglitz) consumption aggregator over these goods:

$$C_{i,t} \equiv \left(\int_{[0,1]} C_{i,t}(\jmath)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}\jmath \right)^{\frac{\varepsilon}{\varepsilon-1}} \quad \text{for} \quad i \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}, \tag{4.1}$$

where $C_{i,t}(\jmath)$ is consumption of good $\jmath \in [0,1]$ per individual of generation i at time t. The parameter ε ($\varepsilon > 1$) is the elasticity of substitution between differentiated goods. The minimum nominal expenditure P_t required to obtain one unit of the composite good and each individuals' expenditure-minimizing demand functions for the differentiated goods are

$$P_t = \left(\int_{[0,1]} P_t(\jmath)^{1-\varepsilon} \mathrm{d}\jmath \right)^{\frac{1}{1-\varepsilon}}, \quad \text{and} \quad C_{i,t}(\jmath) = \left(\frac{P_t(\jmath)}{P_t} \right)^{-\varepsilon} C_{i,t} \quad \text{for all} \quad \jmath \in [0,1], \tag{4.2}$$

where $P_t(\jmath)$ is the nominal price of good $\jmath$, and where the demand functions are conditional on an individual's consumption $C_{i,t}$ of the composite good. An individual's total nominal expenditure on all differentiated goods is

$$\int_{[0,1]} P_t(\jmath) C_{i,t}(\jmath) \mathrm{d}\jmath = P_t C_{i,t}. \tag{4.3}$$

4.2 Firms

There is a measure-one continuum of firms in the economy, each of which has a monopoly on the production and sale of one of the differentiated goods. Each firm is operated by a team of owner-managers who each have an equal claim to the profits of the firm. The participation of a specific team of managers is essential for production, and managers cannot commit to provide input to firms owned by outsiders. In this situation, managers will not be able to sell shares in firms, so the presence of firms does not affect the range of financial assets that can be bought and sold. Firms simply maximize the profits paid out to their owner-managers.

Consider the firm that is the monopoly supplier of good $\jmath$. The firm's output $Y_t(\jmath)$ is subject to the linear production function

$$Y_t(\jmath) = A_t N_t(\jmath), \tag{4.4}$$

where $N_t(\jmath)$ is the number of hours of labour hired by the firm, and A_t is the exogenous level of TFP common to all firms. The firm is a wage taker in the perfectly competitive market for homogeneous labour, where the real wage in units of composite goods is w_t. The real profits of firm $\jmath$, paid out as remuneration to the firm's owner-managers, are:

$$J_t(\jmath) = \frac{P_t(\jmath)}{P_t} Y_t(\jmath) - w_t N_t(\jmath). \tag{4.5}$$

Given the production function [4.4], the real marginal cost of production common to all firms irrespective of their levels of output is

$$x_t = \frac{w_t}{A_t}. \tag{4.6}$$

The firm faces a demand function derived from summing up consumption of good $\jmath$ over all generations (each of which has measure 1/3):

$$\frac{1}{3}C_{\mathrm{y},t}(\jmath) + \frac{1}{3}C_{\mathrm{m},t}(\jmath) + \frac{1}{3}C_{\mathrm{o},t}(\jmath) = Y_t(\jmath). \tag{4.7}$$

Using each individual's demand function [4.2] for good $\jmath$ and the definition [2.2] of aggregate demand C_t for the composite good, the total demand function faced by firm $\jmath$ is

$$Y_t(\jmath) = \left(\frac{P_t(\jmath)}{P_t}\right)^{-\varepsilon} C_t. \tag{4.8}$$

As a monopolist, the firm sets the price $P_t(\jmath)$ of its good. Taking into account the constraints implied by the production function [4.4] and the demand function [4.8], the firm's profits are

$$J_t(\jmath) = \left\{\left(\frac{P_t(\jmath)}{P_t}\right)^{1-\varepsilon} - x_t\left(\frac{P_t(\jmath)}{P_t}\right)^{-\varepsilon}\right\} C_t, \tag{4.9}$$

where the firm takes as given the general price level P_t, real aggregate demand C_t, and real marginal cost x_t (from [4.6]).

At the beginning of time period t, a group of firms is randomly selected to have access to all information available during period t when setting prices. For a firm $\jmath$ among this group, $P_t(\jmath)$ is chosen to maximize the expression for profits $J_t(\jmath)$ in [4.9]. Since the profit function [4.9] is the same across firms, all firms in this group will chose the same price, denoted by $\hat{P}_t$:

$$\frac{\hat{P}_t}{P_t} = \left(\frac{\varepsilon}{\varepsilon - 1}\right) x_t, \tag{4.10}$$

where the term $\varepsilon/(\varepsilon - 1)$ represents each firm's markup of price on marginal cost. The remaining group of firms must set a price in advance of period-t information being revealed (they have access to all information available at the end of period $t-1$, but they are not constrained to use the same price as in the previous period). A firm $\jmath$ in this group chooses $P_t(\jmath)$ to maximize expected profits $\mathbb{E}_{t-1} J_t(\jmath)$. All firms in this group will choose the same nominal price P'_t that satisfies the first-order condition:

$$\mathbb{E}_{t-1}\left[\left(\frac{P'_t}{P_t} - \left(\frac{\varepsilon}{\varepsilon-1}\right) x_t\right)\left(\frac{P'_t}{P_t}\right)^{-\varepsilon} C_t\right] = 0, \quad \text{assuming } \frac{P'_t}{P_t} \geq x_t, \tag{4.11}$$

where the assumption is that the firm will be willing to satisfy whatever level of demand is forthcoming at the preset price in all possible states of the world. Note that the profit-maximization problem has no intertemporal dimension under the assumptions made. Let the parameter κ ($0 < \kappa < \infty$) denote the number of firms in the group with predetermined prices relative to the group who set price with full information.

4.3 Households

An individual born at time t has lifetime utility function [2.1], with the consumption levels $C_{\mathrm{y},t}$, $C_{\mathrm{m},t}$, and $C_{\mathrm{o},t}$ now referring to consumption of the composite good [4.1]. Labour is supplied inelastically, with the number of hours varying over the life cycle. Young, middle-aged, and old individuals respectively supply α_{y}, α_{m}, and α_{o} hours of labour. Individuals also derive income in their role of owner-managers of firms, and it is assumed that the amount of income from this source also varies over the life cycle in the same manner as labour income. Specifically, each young, middle-aged, and old individual belongs respectively to the managerial teams of α_{y}, α_{m}, and α_{o} firms. The total real non-financial incomes of the generations alive at time t are:

$$Y_{\mathrm{y},t} = \alpha_{\mathrm{y}} w_t + \alpha_{\mathrm{y}} J_t, \quad Y_{\mathrm{m},t} = \alpha_{\mathrm{m}} w_t + \alpha_{\mathrm{m}} J_t, \quad \text{and } Y_{\mathrm{o},t} = \alpha_{\mathrm{o}} w_t + \alpha_{\mathrm{o}} J_t, \quad \text{with } J_t \equiv \int_{[0,1]} J_t(\jmath)\mathrm{d}\jmath. \tag{4.12}$$

Individuals receive fixed fractions of total profits J_t because all variation in profits between different firms is owing to the random selection of which firms receive access to full information when setting their prices. The coefficients α_{y}, α_{m}, and α_{o} are parameterized as in [2.4].

The assumptions on financial markets are the same as those considered in section 2. In the benchmark case, there is only a one-period, risk-free, nominal bond as described in section 2.1. It follows that the generational budget constraints are as given in [2.8], where consumption $C_{i,t}$ and income $Y_{i,t}$ are reinterpreted according to [4.1] and [4.12]. The hypothetical case of complete markets can also be considered as in section 2.2, where the generational budget constraints are as in [2.16], with $C_{i,t}$ and $Y_{i,t}$ reinterpreted as described above.

4.4 Equilibrium

The young, middle-aged, and old have per-person labour supplies $H_{\mathrm{y},t} = \alpha_{\mathrm{y}}$, $H_{\mathrm{m},t} = \alpha_{\mathrm{m}}$, and $H_{\mathrm{o},t} = \alpha_{\mathrm{o}}$. Total labour supply is $H_t = (1/3)H_{\mathrm{y},t} + (1/3)H_{\mathrm{m},t} + (1/3)H_{\mathrm{o},t}$, which is fixed at $H_t = 1$ given [2.3]. Equilibrium of the labour market therefore requires

$$\int_{[0,1]} N_t(\jmath)\mathrm{d}\jmath = 1. \tag{4.13}$$

Goods market clearing requires that [4.7] holds for all $\jmath \in [0,1]$, and by using [4.3], this is equivalent to

$$C_t = Y_t, \quad \text{where } Y_t \equiv \int_{[0,1]} \frac{P_t(\jmath)}{P_t} Y_t(\jmath)\mathrm{d}\jmath, \tag{4.14}$$

with Y_t denoting the real value of output summed over all firms. Using [4.5], [4.13], and [4.14], the definition of total profits in [4.12] implies that $J_t = Y_t - w_t$. It follows from [4.12] that $Y_{\mathrm{y},t} = \alpha_{\mathrm{y}} Y_t$, $Y_{\mathrm{m},t} = \alpha_{\mathrm{m}}$, and $Y_{\mathrm{o},t} = \alpha_{\mathrm{o}} Y_t$, as in equation [2.3].

Using the demand functions [4.7] for individual goods and the overall labour- and goods-market equilibrium conditions [4.13] and [4.14]:

$$A_t = \int_{[0,1]} A_t N_t(\jmath)\mathrm{d}\jmath = \int_{[0,1]} Y_t(\jmath)\mathrm{d}\jmath = \left(\int_{[0,1]} \left(\frac{P_t(\jmath)}{P_t}\right)^{-\varepsilon}\right) Y_t, \tag{4.15}$$

which leads to the following aggregate production function:

$$Y_t = \frac{A_t}{\mathit{\Delta}_t}, \quad \text{where } \mathit{\Delta}_t \equiv \left(\int_{[0,1]} \left(\frac{P_t(\jmath)}{P_t}\right)^{-\varepsilon} \mathrm{d}\jmath\right)^{-1}. \tag{4.16}$$

The term $\mathit{\Delta}_t$ represents the effects of misallocation due to relative-price distortions on aggregate productivity.

Let $\hat{p}_t \equiv \hat{P}_t/P_t$ denote the relative price of goods sold by the fraction $1/(1+\kappa)$ of firms that set a price with full information (the parameter κ is the ratio of the number of predetermined-price firms to firms with full information), and $p'_t \equiv P'_t$ the relative price for the fraction $\kappa/(1+\kappa)$ of firms whose price is predetermined. The formula for the price index P_t in [4.2] leads to

$$\frac{1}{1+\kappa}\hat{p}_t^{1-\varepsilon} + \frac{\kappa}{1+\kappa}{p'_t}^{1-\varepsilon} = 1,$$

which implies that $\hat{p}_t$ can be written as a function of p'_t:

$$\hat{p}_t = \left(1 - \kappa\left({p'_t}^{1-\varepsilon} - 1\right)\right)^{\frac{1}{1-\varepsilon}}. \tag{4.17}$$

Equation [4.10] implies that $\hat{p}_t = (\varepsilon/(\varepsilon-1))x_t$, hence by using [4.17], the equation [4.11] for setting the predetermined price becomes:

$$\mathbb{E}_{t-1}\left[\left(p'_t - \left(1-\kappa\left({p'_t}^{1-\varepsilon}-1\right)\right)^{\frac{1}{1-\varepsilon}}\right){p'_t}^{-\varepsilon}Y_t\right] = 0, \quad \text{assuming } \; p'_t \geq (1-\varepsilon^{-1})\left(1-\kappa\left({p'_t}^{1-\varepsilon}-1\right)\right)^{\frac{1}{1-\varepsilon}}. \tag{4.18a}$$

Aggregate output is determined by [4.16]:

$$Y_t = \frac{A_t}{\Delta_t}, \tag{4.18b}$$

and by using [4.17]:

$$\Delta_t = \left(\frac{\kappa}{1+\kappa}{p'_t}^{-\varepsilon} + \frac{1}{1+\kappa}\left(1-\kappa\left({p'_t}^{1-\varepsilon}-1\right)\right)^{-\frac{\varepsilon}{1-\varepsilon}}\right)^{-1}. \tag{4.18c}$$

Finally, note the following definitions:

$$p'_t = \frac{P'_t}{P_t}, \quad \text{with } \; P'_t = \mathbb{E}_{t-1}P'_t, \quad \pi_t = \frac{P_t - P_{t-1}}{P_{t-1}}, \quad \text{and } \; g_t = \frac{Y_t - Y_{t-1}}{Y_{t-1}}. \tag{4.18d}$$

The equilibrium of the model is then given by the solution of equations [2.12], [2.14a]–[2.14b], and [4.18a]–[4.18d], augmented with a monetary policy rule such as [2.6].

Before considering the equilibrium, consider the hypothetical case of fully flexible goods prices ($\kappa = 0$). In this case, [4.18c] implies $\hat{\Delta}_t = 1$, so equilibrium output with flexible prices is $\hat{Y}_t = A_t$, which is simply equal to exogenous TFP. This is also the Pareto-efficient level of aggregate output.

4.5 Steady state and log linearization

In a non-stochastic steady state, [4.18a] implies $\bar{p}' = 1$, and [4.18c] implies $\bar{\Delta} = 1$. Assuming that the steady-state growth rate of A_t is zero then allows Proposition 1 to be applied to determined the steady-state values of the other variables.

The new equations [4.18a]–[4.18d] of the sticky-price model can be log-linearized around this steady state. The remaining equations [2.6], [2.12], [2.14a]–[2.14b] can be log linearized as before in section 3.2.

The log-deviation of the misallocation term Δ_t from [4.18c] is zero up to a first-order approximation ($\mathsf{\Delta}_t = 0$), so it follows that a first-order approximation of aggregate output is

$$\mathsf{Y}_t = \mathsf{A}_t. \tag{4.19}$$

Thus, aggregate output is equal to exogenous TFP up to a first-order approximation. Real GDP growth is thus $\mathsf{g}_t = \mathsf{A}_t - \mathsf{A}_{t-1}$ up to a first-order approximation. For given values of g_t, first-order approximations to the solutions for all other variables can be obtained from the log-linearized equations [3.1] and [3.2a]–[3.2b] using the results in Proposition 2. The hypothetical complete-markets outcome is found by using the results of Proposition 3.

4.6 Optimal monetary policy

With both incomplete financial markets and sticky goods prices, monetary policy has competing objectives. Optimal monetary policy maximizes social welfare [2.22] using the only instrument available to the central bank: control of the nominal interest rate. The Pareto-weights are those that would support the only implementable Pareto-efficient allocation of consumption. Since the weights derived in section 2.3 are conditional on a particular sequence of real GDP growth rates, the growth rate of the Pareto-efficient (flexible price) level of aggregate output is used. It can then be shown how a first-order approximation to the policy that maximizes this welfare function can be found by minimizing a simple quadratic loss function subject to log-linear approximations of the equations describing the economy.

Proposition 9 *Let the Pareto weights $\hat{\omega}_{i,t}$ be those constructed according to equation [2.26] with the assumed rate of GDP growth being $\hat{g}_t = (\hat{Y}_t - \hat{Y}_{t-1})/\hat{Y}_{t-1}$, where $\hat{Y}_t = A_t$ is the Pareto-efficient level of aggregate output, and with $\hat{r}^*_t$ and $\hat{c}^*_{i,t}$ being the complete-markets real return and consumption-GDP ratios derived from [2.20a]–[2.20c] for real GDP growth of $\hat{g}_t$. The following quadratic loss function is equal to the negative of the social welfare function $\mathscr{W}_{t_0}$ from [2.22] using weights $\hat{\omega}_{i,t}$ up to a scaling, with terms independent of monetary policy and terms of third-order and higher suppressed:*

$$\mathscr{L}_{t_0} = \sum_{t=t_0}^{\infty} \beta^{t-t_0} \mathbb{E}_{t_0} \left[\frac{\varepsilon\kappa}{2} (\pi_t - \mathbb{E}_{t-1}\pi_t)^2 + \frac{\gamma^2(\beta^2 + (\theta - \beta)\theta)}{3\sigma} \tilde{\mathsf{d}}_t^2 \right]. \qquad [4.20]$$

The variable $\tilde{\mathsf{d}}_t = \mathsf{d}_t - \mathsf{d}^*_t$ is the 'gap' between the debt/GDP ratio with incomplete and complete financial markets, where the ratio d^*_t with complete markets is characterized in Proposition 3. The coefficient θ is as defined in Proposition 2.

Proof See appendix A.5 ■

The quadratic loss function [4.20] shows that just two variables capture all that needs to be known about the economy's deviation from Pareto efficiency. First, intratemporal misallocation of resources owing to sticky prices is proportional to the square of the inflation surprise $\pi_t - \mathbb{E}_{t-1}\pi_t$. Second, the loss from imperfect risk-sharing in incomplete financial markets is proportional to the square of the debt/GDP ratio. This is analogous to the conventional loss functions seen in optimal monetary policy analyses where there is inflation squared and the output gap squared, where the output gap squared is proportional to the loss from output deviating from its Pareto efficient level. Just as it is output gap fluctuations rather than output fluctuations that are costly in conventional analyses, here it is fluctuations in the debt/GDP gap rather than fluctuations in debt/GDP per se that are costly.

Optimal monetary policy minimizes the quadratic loss function using the nominal interest rate i_t as the instrument, and subject to first-order approximations of the constraints involving the endogenous variables, inflation π_t, and the debt/GDP gap $\tilde{\mathsf{d}}_t$. Using [3.1], [3.12], [3.14] there are two constraints that apply to the three endogenous variables π_t, $\tilde{\mathsf{d}}_t$, and i_t:

$$\lambda\tilde{\mathsf{d}}_t = \mathbb{E}_t\tilde{\mathsf{d}}_{t+1}; \quad [4.21a]$$

$$\pi_t = \mathsf{i}_{t-1} + \theta\tilde{\mathsf{d}}_t + \tilde{\mathsf{d}}_{t-1} - \mathsf{r}_t^*; \quad [4.21b]$$

where r_t^* is determined exogenously using [3.13] and $\mathsf{g}_t = \mathsf{A}_t - \mathsf{A}_{t-1}$. The debt/GDP ratio is then determined as $\mathsf{d}_t = \mathsf{d}_t^* + \tilde{\mathsf{d}}_t$, where d_t^* is as given in [3.11].

Proposition 10 *The optimal monetary policy minimizing the loss function* [4.20] *subject to the constraints* [4.21a]–[4.21b] *is implemented by a weighted nominal income target*

$$\mathcal{N}_t \equiv (1+\varpi)\mathsf{P}_t + \mathsf{Y}_t = \frac{\beta}{2}\mathsf{d}_t^*, \quad \textit{where} \ \ \varpi = \frac{3\sigma(1-\beta\lambda^2)\theta^2\varepsilon\kappa}{2\gamma^2(\beta^2+(\theta-\beta)\theta)}, \quad [4.22]$$

with the level of the target depending on the exogenous complete-markets debt/GDP d_t^* *given in equation* [3.11]. *The price level is over-weighted in calculating the nominal income target.*

Proof Setting up the Lagrangian for the problem of minimizing [4.20] subject to [4.21a] and [4.21b]:

$$\mathcal{L}_{t_0} = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\frac{\varepsilon\kappa}{2}(\pi_t - \mathbb{E}_{t-1}\pi_t)^2 + \frac{\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma}\tilde{\mathsf{d}}_t^2\right]$$
$$+\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\daleth_t\left\{\lambda\tilde{\mathsf{d}}_t - \tilde{\mathsf{d}}_{t+1}\right\} + \beth_t\left\{\mathsf{i}_{t-1} + \theta\tilde{\mathsf{d}}_t + \tilde{\mathsf{d}}_{t-1} - \pi_t - \mathsf{r}_t^*\right\}\right],$$

where the (scaled) Lagrangian multipliers are denoted by $\daleth_t$ and $\beth_t$. The first-order conditions with respect to each of the endogenous variables π_t, $\tilde{\mathsf{d}}_t$, and i_t are:

$$\varepsilon\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t) - \beth_t = 0; \quad [4.23a]$$

$$\frac{2\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma}\tilde{\mathsf{d}}_t + \lambda\daleth_t - \beta^{-1}\daleth_{t-1} + \theta\beth_t + \beta\mathbb{E}_t\beth_{t+1} = 0; \quad [4.23b]$$

$$\mathbb{E}_t\beth_{t+1} = 0. \quad [4.23c]$$

Taking the conditional expectation of equation [4.23b] at time $t+1$, multiplying both sides by β and using [4.23c] to eliminate terms in $\beth_t$:

$$\daleth_t = \beta\lambda\mathbb{E}_t\daleth_{t+1} + \frac{2\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma}\beta\mathbb{E}_t\tilde{\mathsf{d}}_{t+1}.$$

Solving forwards using this equation and noting that [4.21a] implies $\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell} = \lambda^\ell\tilde{\mathsf{d}}_t$:

$$\daleth_t = \frac{2\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma}\beta\sum_{\ell=1}^{\infty}(\beta\lambda)^{\ell-1}\mathbb{E}_t\tilde{\mathsf{d}}_{t+\ell} = \frac{2\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma}\frac{\beta\lambda\tilde{\mathsf{d}}_t}{1-\beta\lambda^2}.$$

The expression for $\daleth_t$ in the equation above implies that:

$$\frac{2\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma}\tilde{d}_t + \lambda\daleth_t - \beta^{-1}\daleth_{t-1} = \frac{2\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma(1-\beta\lambda^2)}(\tilde{d}_t - \lambda\tilde{d}_{t-1}).$$

Using the equation above and the formula for $\beth_t$ that follows from [4.23a] and substituting this into [4.23b] yields:

$$\frac{2\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma(1-\beta\lambda^2)}(\tilde{d}_t - \mathbb{E}_{t-1}\tilde{d}_t) + \varepsilon\kappa\theta(\pi_t - \mathbb{E}_{t-1}\pi_t) = 0,$$

where [4.23c] has again been used to eliminate $\mathbb{E}_t\beth_{t+1}$, and [4.21a] to replace $\lambda\tilde{d}_{t-1}$ by $\mathbb{E}_{t-1}\tilde{d}_t$. This equation can be solved for the unexpected component of the debt gap:

$$\tilde{d}_t - \mathbb{E}_{t-1}\tilde{d}_t = -\frac{3\sigma(1-\beta\lambda^2)\theta\varepsilon\kappa}{2\gamma^2(\beta^2+(\theta-\beta)\theta)}(\pi_t - \mathbb{E}_{t-1}\pi_t).$$

Equation [4.21b] implies $\pi_t - \mathbb{E}_{t-1}\pi_t = \theta(\tilde{d}_t - \mathbb{E}_{t-1}\tilde{d}_t) - (r^*_t - \mathbb{E}_{t-1}r^*_t)$, and by substituting the equation above:

$$(1+\varpi)(\pi_t - \mathbb{E}_{t-1}\pi_t) + (r^*_t - \mathbb{E}_{t-1}r^*_t) = 0, \qquad [4.24]$$

where ϖ is as defined in [4.22].

Now first-difference both sides of the weighted nominal income target [4.22] to obtain $(1+\varpi)\pi_t + g_t = (\beta/2)(d^*_t - d^*_{t-1})$. Equating the unexpected components of both sides of this equation yields:

$$(1+\varpi)(\pi_t - \mathbb{E}_{t-1}\pi_t) + (g_t - \mathbb{E}_{t-1}g_t) = \frac{\beta}{2}(d^*_t - \mathbb{E}_{t-1}d^*_t). \qquad [4.25]$$

Comparison of [3.11] and [3.13] implies that $r^*_t - \mathbb{E}_{t-1}r^*_t = (g_t - \mathbb{E}_{t-1}g_t) - (\beta/2)(d^*_t - \mathbb{E}_{t-1}d^*_t)$. Therefore, equation [4.25] is equivalent to [4.24], so the proposed weighted nominal income target [4.22] implements the optimal monetary policy. ■

5 Endogenous labour supply

In the analysis of section 4, neither monetary policy nor the incompleteness of markets had any first-order effect on aggregate output. This section adds an endogenous labour supply decision, which will imply that both monetary policy and the incompleteness of financial markets have consequences for aggregate output. The analysis will further demonstrate that the desire to stabilize the debt/GDP ratio with nominal income targeting is present even if the policymaker does not care about risk sharing.

5.1 Households

The population and age structure of households is the same as that described in section 2, but the lifetime utility function of individuals born at time t is now

$$\mathscr{U}_t = \left\{\log C_{\mathrm{y},t} - \frac{H_{\mathrm{y},t}^{\eta}}{\eta\alpha_{\mathrm{y}}^{\eta-1}}\right\} + \beta\mathbb{E}_t\left\{\log C_{\mathrm{m},t+1} - \frac{H_{\mathrm{m},t+1}^{\eta}}{\eta\alpha_{\mathrm{m}}^{\eta-1}}\right\} + \beta^2\mathbb{E}_t\left\{\log C_{\mathrm{o},t+2} - \frac{H_{\mathrm{o},t+2}^{\eta}}{\eta\alpha_{\mathrm{o}}^{\eta-1}}\right\}, \qquad [5.1]$$

where $H_{\mathrm{y},t}$, $H_{\mathrm{m},t}$, and $H_{\mathrm{o},t}$ are respectively the per-person hours of labour supplied by young, middle-aged, and old individuals at time t. The utility function is additively separable between consumption and hours, and utility is logarithmic in consumption (an intertemporal elasticity of substitution σ of unity), with the composite consumption good being [4.1], as in section 4. The parameter η ($1 < \eta < \infty$) is related to the Frisch elasticity of labour supply, the Frisch elasticity being $(\eta - 1)^{-1}$. The parameters α_{y}, α_{m}, and α_{o}, which in section 2 specified the shares of the exogenous income endowment received by each generation, are now interpreted as age-specific differences in the disutility of working. A higher value of α_i indicates that generation $i \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}$ has a relatively low disutility of labour.

Hours of labour supplied by individuals of different ages are not perfect substitutes, so wages are age specific. Let $w_{\mathrm{y},t}$, $w_{\mathrm{m},t}$, and $w_{\mathrm{o},t}$ denote the hourly (real) wages of the young, middle-aged, and old, respectively. As in section 4, individuals earn remuneration as owner-managers of firms. Managerial labour is assumed to have no disutility and is supplied inelastically, with α_{y}, α_{m}, and α_{o} denoting the per-person proportions of total profits J_t received by individuals of each generation. Individuals are also subject to age-specific lump-sum taxes $T_{\mathrm{y},t}$, $T_{\mathrm{m},t}$, and $T_{\mathrm{o},t}$. The per-person real non-financial incomes of individuals from different generations are:

$$Y_{\mathrm{y},t} = w_{\mathrm{y},t}H_{\mathrm{y},t} + \alpha_{\mathrm{y}}J_t - T_{\mathrm{y},t}, \quad Y_{\mathrm{m},t} = w_{\mathrm{m},t}H_{\mathrm{m},t} + \alpha_{\mathrm{m}}J_t - T_{\mathrm{m},t}, \quad \text{and} \quad Y_{\mathrm{o},t} = w_{\mathrm{y},t}H_{\mathrm{y},t} + \alpha_{\mathrm{y}}J_t - T_{\mathrm{y},t}, \qquad [5.2]$$

where total profits J_t are as defined in [4.12].

Given additive separability of the utility function between consumption and hours, the consumption Euler equations for each generation are the same (after setting $\sigma = 1$) as those in [2.9] in the case of incomplete markets, and [2.17] in the hypothetical case of complete markets. Irrespective of the assumptions on financial markets, the optimality conditions for maximizing utility [5.1] with respect to labour supply $H_{i,t}$ subject to [5.2] and the appropriate budget constraint are:

$$C_{\mathrm{y},t}\left(\frac{H_{\mathrm{y},t}}{\alpha_{\mathrm{y}}}\right)^{\eta-1} = w_{\mathrm{y},t}, \quad C_{\mathrm{m},t}\left(\frac{H_{\mathrm{m},t}}{\alpha_{\mathrm{m}}}\right)^{\eta-1} = w_{\mathrm{m},t}, \quad \text{and} \quad C_{\mathrm{o},t}\left(\frac{H_{\mathrm{o},t}}{\alpha_{\mathrm{o}}}\right)^{\eta-1} = w_{\mathrm{o},t}. \qquad [5.3]$$

5.2 Firms

There is a range of differentiated goods and monopolistically competitive firms as in the model of section 4. The production function is [4.4], but now the labour $N_t(\jmath)$ used by firm $\jmath$ is reinterpreted as a composite labour input drawing on hours of labour from young, middle-aged, and old workers.

The labour aggregator is assumed to have the Cobb-Douglas form:

$$N_t(\jmath) \equiv \frac{N_{\mathrm{y},t}(\jmath)^{\frac{\alpha_{\mathrm{y}}}{3}} N_{\mathrm{m},t}(\jmath)^{\frac{\alpha_{\mathrm{m}}}{3}} N_{\mathrm{o},t}(\jmath)^{\frac{\alpha_{\mathrm{o}}}{3}}}{3\alpha_{\mathrm{y}}^{\frac{\alpha_{\mathrm{y}}}{3}} \alpha_{\mathrm{m}}^{\frac{\alpha_{\mathrm{m}}}{3}} \alpha_{\mathrm{o}}^{\frac{\alpha_{\mathrm{o}}}{3}}}, \qquad [5.4]$$

where $N_{i,t}(\jmath)$ is the firm's employment of hours of labour by individuals of age $i \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}$, and α_{y}, α_{m}, and α_{o} are the same parameters that appear in [5.1]. The Cobb-Douglas form of [5.4] implies an elasticity of substitution between different labour inputs of one.

The firm takes the real wages $w_{\mathrm{y},t}$, $w_{\mathrm{m},t}$, and $w_{\mathrm{o},t}$ in each age-specific labour market as given. Firms are assumed to receive a proportional wage-bill subsidy of τ from the government. Irrespective of whether or not firms set prices with full information, firms choose labour inputs $N_{i,t}(\jmath)$ to minimize the total cost of obtaining the number of units $N_t(\jmath)$ of composite labour that enter the production function [4.4] subject to the aggregator [5.4]. Conditional on $N_t(\jmath)$, the cost-minimizing labour demand functions are:

$$N_{\mathrm{y},t}(\jmath) = \frac{\alpha_{\mathrm{y}} w_t}{3(1-\tau) w_{\mathrm{y},t}} N_t(\jmath), \quad N_{\mathrm{m},t}(\jmath) = \frac{\alpha_{\mathrm{m}} w_t}{3(1-\tau) w_{\mathrm{m},t}} N_t(\jmath), \quad \text{and} \ N_{\mathrm{o},t}(\jmath) = \frac{\alpha_{\mathrm{o}} w_t}{3(1-\tau) w_{\mathrm{o},t}} N_t(\jmath), \qquad [5.5]$$

where w_t is the minimized value of (post-subsidy) wage costs per unit of composite labour:

$$w_t = (1-\tau) w_{\mathrm{y},t}^{\frac{\alpha_{\mathrm{y}}}{3}} w_{\mathrm{m},t}^{\frac{\alpha_{\mathrm{m}}}{3}} w_{\mathrm{o},t}^{\frac{\alpha_{\mathrm{o}}}{3}}, \quad \text{and where} \ (1-\tau)\left(w_{\mathrm{y},t} N_{\mathrm{y},t}(\jmath) + w_{\mathrm{m},t} N_{\mathrm{m},t}(\jmath) + w_{\mathrm{o},t} N_{\mathrm{o},t}(\jmath)\right) = w_t N_t(\jmath). \qquad [5.6]$$

All the equations in section 4.2 continue to hold with w_t as given in [5.6], so firms can be analysed as if they were directly hiring units of composite labour at real wage w_t for use in the production function [4.4].

5.3 Government

The only role of government in the model is to raise lump-sum taxes to fund the wage-bill subsidy paid to firms. The government does not spend, borrow, or save. The total cost of the wage-bill subsidy is:

$$T_t = \tau \int_{[0,1]} \left(w_{\mathrm{y},t} N_{\mathrm{y},t}(\jmath) + w_{\mathrm{m},t} N_{\mathrm{m},t}(\jmath) + w_{\mathrm{o},t} N_{\mathrm{o},t}(\jmath)\right) \mathrm{d}\jmath. \qquad [5.7]$$

The subsidy is set so that $\tau = 1/\varepsilon$, where ε is the elasticity of substitution between different goods in [4.1] (this implies a well-defined subsidy because $\varepsilon > 1$). The government distributes the tax burden so that the per-person age-specific tax levels are:

$$T_{\mathrm{y},t} = \alpha_{\mathrm{y}} T_t, \quad T_{\mathrm{m},t} = \alpha_{\mathrm{m}} T_t, \quad \text{and} \ T_{\mathrm{o},t} = \alpha_{\mathrm{o}} T_t, \qquad [5.8]$$

where α_{y}, α_{m}, and α_{o} are the same parameters as in [5.1] and [5.4].

5.4 Equilibrium

There is a separate market-clearing condition for each age-specific labour market:

$$\int_{[0,1]} N_{i,t}(\jmath)\mathrm{d}\jmath = \frac{1}{3}H_{i,t} \quad \text{for each} \quad i \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}. \tag{5.9}$$

Following the same steps as in [4.15] using [4.4] and [4.7] leads to the aggregate production function

$$Y_t = \frac{A_t N_t}{\Delta_t}, \quad \text{with} \quad N_t \equiv \int_{[0,1]} N_t(\jmath)\mathrm{d}\jmath, \tag{5.10}$$

where N_t denotes the aggregate usage of units of composite labour, and Δ_t is the measure of relative-price distortions defined in [4.16].

Using the cost-minimizing labour demand functions [5.5], the market clearing condition [5.9], and the definition of aggregate demand N_t for units of composite labour:

$$w_{i,t}H_{i,t} = \alpha_i \left(\frac{w_t N_t}{1-\tau}\right) \quad \text{for each} \quad i \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}. \tag{5.11}$$

Using [4.5], [4.14], [4.12], and [5.10], total profits are given by $J_t = Y_t - w_t N_t$. Using [5.6], [5.7], and [5.10], total taxes are $T_t = (\tau/(1-\tau))w_t N_t$. With [5.8], age-specific non-financial incomes from [5.2] are $Y_{i,t} = w_{i,t}H_{i,t} + \alpha_i(J_t - T_t)$ for each i. Together with the expressions for J_t and T_t, equation [5.11] implies that $Y_{i,t} = \alpha_i Y_t$, thus that the age-shares of total GDP are constant.

Substituting [5.11] into the labour supply first-order condition [5.3] implies that the age-specific real wages $w_{i,t}$ satisfy:

$$w_{i,t} = c_{i,t}Y_t \left(\frac{w_t N_t}{(1-\tau)w_{i,t}}\right)^{\eta-1} \quad \text{for each} \quad i \in \{\mathrm{y}, \mathrm{m}, \mathrm{o}\}, \tag{5.12}$$

where $c_{i,t} \equiv C_{i,t}/Y_t$. Then using the formula [5.6] for the minimum cost w_t of a unit of composite labour and the size of the wage-bill subsidy $\tau = \varepsilon^{-1}$ yields:

$$w_t = (1-\varepsilon^{-1}) \left(c_{\mathrm{y},t}^{\frac{\alpha_\mathrm{y}}{3}} c_{\mathrm{m},t}^{\frac{\alpha_\mathrm{m}}{3}} c_{\mathrm{o},t}^{\frac{\alpha_\mathrm{o}}{3}}\right) N_t^{\eta-1} Y_t.$$

Substituting this into equation [4.6] for real marginal cost x_t and using [5.10] to replace N_t:

$$x_t = (1-\varepsilon^{-1}) \left(c_{\mathrm{y},t}^{\frac{\alpha_\mathrm{y}}{3}} c_{\mathrm{m},t}^{\frac{\alpha_\mathrm{m}}{3}} c_{\mathrm{o},t}^{\frac{\alpha_\mathrm{o}}{3}}\right) \Delta_t^{\eta-1} \left(\frac{Y_t}{A_t}\right)^{\eta}. \tag{5.13}$$

Since [4.10] implies $(1-\varepsilon^{-1})\hat{p}_t = x_t$, combining [4.17] and [5.13] implies:

$$\left(1 - \kappa\left({p_t'}^{1-\varepsilon} - 1\right)\right)^{\frac{1}{1-\varepsilon}} = \left(c_{\mathrm{y},t}^{\frac{\alpha_\mathrm{y}}{3}} c_{\mathrm{m},t}^{\frac{\alpha_\mathrm{m}}{3}} c_{\mathrm{o},t}^{\frac{\alpha_\mathrm{o}}{3}}\right) \Delta_t^{\eta-1} \left(\frac{Y_t}{A_t}\right)^{\eta}. \tag{5.14}$$

With incomplete markets, the equilibrium is determined by equations [2.12], [2.14a]–[2.14b], [4.18a], [4.18c], [4.18d], [5.14], augmented with a monetary policy rule such as [2.6].

5.5 Steady state and log linearization

Equation [4.18a] implies $\bar{p}' = 1$, and [4.18c] implies $\bar{\Delta} = 1$ as in section 4. If the steady-state growth rate of A_t is zero as assumed then from [5.14] the only possible steady-state growth rate of Y_t must also be zero. Given this, Proposition 1 demonstrates that $\bar{c}_{\mathrm{y}} = \bar{c}_{\mathrm{m}} = \bar{c}_{\mathrm{o}} = 1$, which is consistent with equation [5.14].

The log-linearization of equations [2.12], [2.14a]–[2.14b], [4.18c], and [4.18d] are as in section 3.2 and section 4. With logarithmic utility in consumption ($\sigma = 1$), Proposition 3 implies that $\mathsf{d}_t^* = 0$ and $\mathsf{r}_t^* = \mathsf{g}_t$, and hence that $\tilde{\mathsf{d}}_t = \mathsf{d}_t$.

Equation [4.18a] for the predetermined price becomes $\mathbb{E}_{t-1}\mathsf{p}_t' = 0$ when log linearized. With $\mathsf{p}_t' = \mathsf{P}_t' - \mathsf{P}_t$ and $\mathsf{P}_t' = \mathbb{E}_{t-1}\mathsf{P}_t'$, and $\pi_t = \mathsf{P}_t - \mathsf{P}_{t-1}$ from [4.18d], this implies $\mathsf{P}_t' = \mathbb{E}_{t-1}\mathsf{P}_t$ and $\mathsf{p}_t' = -(\mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t)$, hence:

$$\mathsf{p}_t' = -(\pi_t - \mathbb{E}_{t-1}\pi_t). \qquad [5.15]$$

The log linearization of equation [5.14] is

$$-\kappa\mathsf{p}_t' = \eta(\mathsf{Y}_t - \mathsf{A}_t) + \left(\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t} + \frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t} + \frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t}\right), \qquad [5.16]$$

noting that $\Delta_t = 0$ up to a first-order approximation.

In the hypothetical case of complete markets with $\sigma = 1$, equation [3.10a] implies $\mathsf{c}_{\mathrm{y},t}^* = \mathsf{c}_{\mathrm{m},t}^* = \mathsf{c}_{\mathrm{o},t}^* = 0$ using $\mathsf{d}_t^* = 0$ and $\mathsf{r}_t^* = \mathsf{g}_t$. Thus, equation [5.16] implies the Pareto-efficient level of aggregate output $\hat{\mathsf{Y}}_t^*$ that prevails with both complete financial markets and fully flexible prices ($\kappa = 0$) is $\hat{\mathsf{Y}}_t^* = \mathsf{A}_t$. Letting $\tilde{\mathsf{Y}}_t \equiv \mathsf{Y}_t - \hat{\mathsf{Y}}_t^*$ denote the output gap, the combination of equations [5.15] and [5.16] implies

$$\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t) = \eta\tilde{\mathsf{Y}}_t + \left(\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t} + \frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t} + \frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t}\right). \qquad [5.17]$$

With $\tilde{\mathsf{d}}_t = \mathsf{d}_t$, $\mathsf{r}_t^* = \mathsf{Y}_t - \mathsf{Y}_{t-1}$, and the definition of the output gap $\tilde{\mathsf{Y}}_t \equiv \mathsf{Y}_t - \hat{\mathsf{Y}}_t^*$, equation [3.14] with [3.1] implies:

$$\pi_t = \mathsf{i}_{t-1} + \theta\mathsf{d}_t + \mathsf{d}_{t-1} - \tilde{\mathsf{Y}}_t + \tilde{\mathsf{Y}}_{t-1} - (\hat{\mathsf{Y}}_t^* - \hat{\mathsf{Y}}_{t-1}^*). \qquad [5.18]$$

Finally, note that with $\mathsf{c}_{\mathrm{y},t} = \tilde{\mathsf{c}}_{\mathrm{y},t}$, $\mathsf{c}_{\mathrm{m},t} = \tilde{\mathsf{c}}_{\mathrm{m},t}$, and $\mathsf{c}_{\mathrm{o},t} = \tilde{\mathsf{c}}_{\mathrm{o},t}$, equation [3.15] implies:

$$\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t} + \frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t} + \frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t} = \xi\mathsf{d}_t, \quad \text{where } \xi = \frac{\gamma^2}{3}\left((2+\beta)\theta - (1+2\beta)\beta\right), \qquad [5.19]$$

using the parameterization of α_{y}, α_{m}, and α_{o} in [2.4]. Therefore, the constraints faced by the policymaker are given by [3.12] with $\mathsf{d}_t = \tilde{\mathsf{d}}_t$, equation [5.18] with $\hat{\mathsf{r}}_t^* = \hat{\mathsf{g}}_t^* = \hat{\mathsf{Y}}_t^* - \hat{\mathsf{Y}}_{t-1}^*$ denoting the hypothetical flexible-price complete-markets real return, and the combination of equations [5.17] and [5.19]:

$$\lambda\mathsf{d}_t = \mathbb{E}_t\mathsf{d}_{t+1}; \qquad [5.20a]$$

$$\pi_t = \mathsf{i}_{t-1} + \theta\mathsf{d}_t + \mathsf{d}_{t-1} - \tilde{\mathsf{Y}}_t + \tilde{\mathsf{Y}}_{t-1} - \hat{\mathsf{r}}_t^*; \qquad [5.20b]$$

$$\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t) = \eta\tilde{\mathsf{Y}}_t + \xi\mathsf{d}_t. \qquad [5.20c]$$

There are four endogenous variables: inflation π_t, the debt/GDP ratio d_t, the output gap $\tilde{\mathsf{Y}}_t$, and the nominal interest rate i_t; and one exogenous variable $\hat{\mathsf{r}}^*_t = \mathsf{A}_t - \mathsf{A}_{t-1}$, which depends only on the growth rate of exogenous TFP A_t.

5.6 Optimal monetary policy

Optimal monetary policy maximizes the social welfare function:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0}\left[\frac{1}{3}\sum_{t=t_0-2}^{\infty}\mathscr{U}_t\right]. \qquad [5.21]$$

This is the social welfare function in [2.22] with constant weights, which is appropriate for the case of $\sigma = 1$ given the results of Proposition 3.

Proposition 11 *The social welfare function* [5.21] *can be approximated by the loss function:*

$$\mathscr{L}_{t_0} = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\frac{\varepsilon\kappa}{2}(\pi_t - \mathbb{E}_{t-1}\pi_t)^2 + \frac{\eta}{2}\tilde{\mathsf{Y}}_t^2 + \frac{\chi}{2}\mathsf{d}_t^2\right], \qquad [5.22]$$

where the weight χ *on the debt/GDP ratio is:*

$$\chi = \frac{\gamma^2}{3}\left(2\left(1+\frac{1}{\eta}\right)((\theta-\beta)\theta+\beta^2) + \frac{\beta\gamma}{\eta}\left(\theta^2 - 2(1+\beta)\theta+\beta\right) - \frac{\gamma^2}{3\eta}\left((2+\beta)\theta-(1+2\beta)\beta\right)^2\right). \qquad [5.23]$$

Optimal monetary policy is a weighted nominal income target:

$$\mathcal{N}_t \equiv (1+\varpi)\mathsf{P}_t + \mathsf{Y}_t, \quad \textit{where}\ \ \varpi = \frac{\kappa\theta\left(\theta\left(\varepsilon+\frac{\kappa}{\eta}\right)+(\varepsilon-1)\frac{\xi}{\eta}\right)}{\left(\chi+\frac{\xi^2}{\eta}\right)\left(1+\left(1+\frac{\kappa}{\eta}\right)\left(\frac{\beta\lambda^2}{1-\beta\lambda^2}\right)\right)+\frac{\kappa}{\eta}(\chi-\xi\theta)}. \qquad [5.24]$$

PROOF See appendix A.6 ■

6 Conclusions

This paper has shown how a monetary policy of nominal GDP targeting facilitates risk sharing in incomplete financial markets where contracts are denominated in terms of money. In an environment where risk derives from uncertainty about future real GDP, strict inflation targeting would lead to a very uneven distribution of risk, with leveraged borrowers' consumption highly exposed to any unexpected change in their incomes when monetary policy prevents any adjustment of the real value of their liabilities. This concentration of risk implies that volumes of credit, long-term real interest rates, and asset prices would be excessively volatile. Strict inflation targeting does provide savers with a risk-free real return, but fundamentally, the economy lacks any technology that delivers risk-free real returns, so the safety of savers' portfolios is simply the flip-side of borrowers' leverage and high levels of risk. Absent any changes in the physical investment technology available to the economy, risk cannot be annihilated, only redistributed.

That leaves the question of whether the distribution of risk is efficient. The combination of incomplete markets and strict inflation targeting implies a particularly inefficient distribution of risk. If complete financial markets were available, borrowers would issue state-contingent debt where the contractual repayment is lower in a recession and higher in a boom. These securities would resemble equity shares in GDP, and they would have the effect of reducing the leverage of borrowers and hence distributing risk more evenly. In the absence of such financial markets, in particular because of the inability of individuals to short-sell such securities, a monetary policy of nominal GDP targeting can effectively complete the market even when only non-contingent nominal debt is available. Nominal GDP targeting operates by stabilizing the debt-to-GDP ratio. With financial contracts specifying liabilities fixed in terms of money, a policy that stabilizes the monetary value of real incomes ensures that borrowers are not forced to bear too much of the aggregate risk, converting nominal debt into real equity.

While the model is far too simple to apply to the recent financial crises and deep recessions experienced by a number of economies, one policy implication does resonate with the predicament of several economies faced with high levels of debt combined with stagnant or falling GDPs. Nominal GDP targeting is equivalent to a countercyclical price level, so the model suggests that higher inflation can be optimal in recessions. In other words, while each of the 'stagnation' and 'inflation' that make up the word 'stagflation' is bad in itself, if stagnation cannot immediately be remedied, some inflation might be a good idea to compensate for the inefficiency of incomplete financial markets. And even if policymakers were reluctant to abandon inflation targeting, the model does suggest that they have the strongest incentives to avoid deflation during recessions (a procyclical price level). Deflation would raise the real value of debt, which combined with falling real incomes would be the very opposite of the risk sharing stressed in this paper, and even worse than an unchanging inflation rate.

It is important to stress that the policy implications of the model in recessions are matched by equal and opposite prescriptions during an expansion. Thus, it is not just that optimal monetary policy tolerates higher inflation in a recession — it also requires lower inflation or even deflation during a period of high growth. Pursuing higher inflation in recessions without following a symmetric policy during an expansion is both inefficient and jeopardizes an environment of low inflation on average. Therefore the model also argues that more should be done by central banks to 'take away the punch bowl' during a boom even were inflation to be stable.

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A Technical appendix

A.1 Proof of Proposition 1

In the non-stochastic steady state, real GDP growth is zero: $\bar{g} = 0$. The steady-state budget constraints with incomplete markets follow immediately from [2.14a] with the specification of the lifecycle income profile [2.4]:

$$\bar{c}_{\text{y}} = (1 - \beta\gamma) + \bar{d}, \quad \bar{c}_{\text{m}} + \bar{d} = (1 + (1 + \beta)\gamma) - (1 + \bar{r})\bar{d}, \quad \text{and} \quad \bar{c}_{\text{o}} = (1 - \gamma) + (1 + \bar{r})\bar{d}. \qquad \text{[A.1.1a]}$$

Using the definition of β in [2.1], the Euler equations [2.14b] in steady state become:

$$\bar{c}_{\text{m}} = \left(\frac{1 + \bar{r}}{1 + \varrho}\right)^{\sigma} \bar{c}_{\text{y}}, \quad \text{and} \quad \bar{c}_{\text{o}} = \left(\frac{1 + \bar{r}}{1 + \varrho}\right)^{\sigma} \bar{c}_{\text{m}}, \qquad \text{[A.1.1b]}$$

where ϱ is the rate of time preference. The steady-state goods market clearing condition [2.5] is

$$\frac{1}{3}\bar{c}_{\text{y}} + \frac{1}{3}\bar{c}_{\text{m}} + \frac{1}{3}\bar{c}_{\text{o}} = 1. \qquad \text{[A.1.1c]}$$

The steady-state Fisher equation [3.6] is:

$$1 + \bar{r} = \frac{1 + \bar{i}}{1 + \bar{\pi}}, \qquad \text{[A.1.1d]}$$

and the monetary policy rule [2.6] is:

$$1 + \bar{i} = \psi_0(1 + \bar{\pi})^{\psi_\pi}. \qquad \text{[A.1.1e]}$$

In the case of complete markets, the steady-state versions of [2.20a] and [2.20b] are the same as [A.1.1a]–[A.1.1b], and any solution satisfying [A.1.1b] is consistent with [2.20c] in steady state. Thus, the set of non-stochastic steady states is the same whether or not markets are complete, hence $\bar{d} = \bar{d}^*$ and $\bar{r} = \bar{r}^*$, and so on, in what follows.

The system of equations [A.1.1a]–[A.1.1c] is block recursive in the real variables $\bar{c}_y$, $\bar{c}_m$, $\bar{c}_o$, $\bar{r}$, and $\bar{d}$, with one equation being redundant. Thus, the steady-state for real variables is invariant to monetary policy.

The budget constraints of the young and old in [A.1.1a] can be used to obtain and equation in $\bar{c}_y$ and $\bar{c}_o$, eliminating $\bar{d}$:

$$(1+\bar{r})\left(\bar{c}_y - (1-\beta\gamma)\right) = (1+\bar{r})\bar{d} = \left(\bar{c}_o - (1-\gamma)\right), \qquad \text{[A.1.2]}$$

and then dividing both sides by $(1+\varrho)$:

$$\left(\frac{1+\bar{r}}{1+\varrho}\right)\left(\bar{c}_y - (1-\beta\gamma)\right) = \beta\left(\bar{c}_o - (1-\gamma)\right). \qquad \text{[A.1.3]}$$

Now define a variable z as follows

$$z \equiv \left(\frac{1+\bar{r}}{1+\varrho}\right)^{\sigma}, \quad \text{noting that } \bar{c}_m = z\bar{c}_y, \quad \bar{c}_o = z\bar{c}_m, \quad \text{and } \bar{c}_o = z^2\bar{c}_y, \qquad \text{[A.1.4]}$$

using the Euler equations in [A.1.1b]. With economically meaningful $\bar{r}$ in the range $-1 < \bar{r} < \infty$ and the parameters satisfying $0 < \varrho < \infty$ and $\sigma > 0$, the range of economically meaningful z values is $0 < z < \infty$.

Substituting the equations from [A.1.4] into the goods market clearing condition [A.1.1c] yields expressions for age-specific consumption in terms of z:

$$\bar{c}_y = \frac{3}{1+z+z^2}, \quad \bar{c}_m = \frac{3z}{1+z+z^2}, \quad \text{and } \bar{c}_o = \frac{3z^2}{1+z+z^2}. \qquad \text{[A.1.5]}$$

Substituting the expressions from [A.1.5] into [A.1.3] and using the definition of z to replace the term $(1+\bar{r})/(1+\varrho)$:

$$\left(\frac{3}{1+z+z^2} - (1-\beta\gamma)\right) z^{\frac{1}{\sigma}} = \beta\left(\frac{3z^2}{1+z+z^2} - (1-\gamma)\right).$$

Multiplying both sides by $-(1+z+z^2)$ yields an equivalent equation:

$$\left((1-\beta\gamma)(1+z+z^2) - 3\right) z^{\frac{1}{\sigma}} = \beta\left((1-\gamma)(1+z+z^2) - 3z^2\right). \qquad \text{[A.1.6]}$$

Define the function $\mathscr{F}(z)$ as follows:

$$\mathscr{F}(z) \equiv \left((1-\beta\gamma)(1+z+z^2) - 3\right) z^{\frac{1}{\sigma}} + \beta\left(3z^2 - (1-\gamma)(1+z+z^2)\right). \qquad \text{[A.1.7]}$$

Comparison with [A.1.6] shows that a necessary and sufficient condition for a steady state is the equation $\mathscr{F}(z) = 0$, with the implied value of $\bar{r}$ found using [A.1.4], the values of $\bar{c}_y$, $\bar{c}_m$, and $\bar{c}_o$ found using [A.1.5], and $\bar{d}$ from [A.1.2]. Note that the function $\mathscr{F}(z)$ is continuous and differentiable for all $z \in (0, \infty)$.

(i) A steady state always exists. Using [A.1.7]

$$\mathscr{F}(1) = 3((1-\beta\gamma) - 1) + 3\beta(1-(1-\gamma)) = 0,$$

hence $z = 1$ is a steady state. The definition of z in [A.1.4] implies $\bar{r} = \varrho$ since $\sigma > 0$. From [A.1.5] it follows that $\bar{c}_y = \bar{c}_m = \bar{c}_o = 1$. Finally, [A.1.2] implies $\bar{d} = \gamma/(1+\bar{r})$. Since $\bar{r} = \varrho$ and $\beta = 1/(1+\varrho)$, this means that $\bar{d} = \beta\gamma$.

(ii) There are both necessary and sufficient conditions to consider.

Necessary conditions

Using [A.1.7], observe that

$$\mathscr{F}(0) = -\beta(1-\gamma) < 0, \quad \text{and} \quad \lim_{z\to\infty} \mathscr{F}(z) = \infty, \qquad \text{[A.1.8]}$$

where the latter statement follows because the highest power of z in [A.1.7] has coefficient $(1-\beta\gamma)$, which is positive. Now note that the derivative of $\mathscr{F}(z)$ from [A.1.7] is

$$\mathscr{F}'(z) = (1-\beta\gamma)(1+2z)z^{\frac{1}{\sigma}} + \frac{1}{\sigma}\left((1-\beta\gamma)(1+z+z^2)-3\right)z^{\frac{1}{\sigma}-1} + \beta\left(6z-(1-\gamma)(1+2z)\right). \qquad \text{[A.1.9]}$$

Evaluating this derivative at $z = 1$:

$$\begin{aligned}\mathscr{F}'(1) = 3(1-\beta\gamma) + \frac{3}{\sigma}\left((1-\beta\gamma)-1\right) + 3\beta\left(2-(1-\gamma)\right) &= 3\left(1-\beta\gamma-\frac{\beta\gamma}{\sigma}+\beta+\beta\gamma\right) \\ &= 3(1+\beta)\left(1-\frac{\beta}{1+\beta}\frac{\gamma}{\sigma}\right).\end{aligned} \qquad \text{[A.1.10]}$$

Using the definition of β from [2.1], the necessary condition stated in the proposition is

$$\frac{\gamma}{\sigma} < \frac{1+\beta}{\beta}. \qquad \text{[A.1.11]}$$

If $\gamma/\sigma > (1+\beta)/\beta$ then [A.1.10] implies that $\mathscr{F}'(1) < 0$. Since $\mathscr{F}(1) = 0$, this means that $\mathscr{F}(z)$ is strictly positive in a neighbourhood below $z = 1$, and strictly negative in a neighbourhood above $z = 1$. Given the findings in [A.1.8] and the continuity of $\mathscr{F}(z)$, it follows that $\mathscr{F}(z) = 0$ has solutions in the economically meaningful ranges $(0, 1)$ and $(1, \infty)$. The steady state $z = 1$ would not then be unique.

To analyse the case $\gamma/\sigma = (1+\beta)/\beta$, use [A.1.9] to obtain the second derivative of $\mathscr{F}(z)$:

$$\begin{aligned}\mathscr{F}''(z) = 2(1-\beta\gamma)z^{\frac{1}{\sigma}} &+ \frac{1}{\sigma}(1-\beta\gamma)(1+2z)z^{\frac{1}{\sigma}-1} + \frac{1}{\sigma}(1-\beta\gamma)(1+2z)z^{\frac{1}{\sigma}-1} \\ &+ \frac{1}{\sigma}\left(\frac{1}{\sigma}-1\right)\left((1-\beta\gamma)(1+z+z^2)-3\right)z^{\frac{1}{\sigma}-2} + \beta\left(6-2(1-\gamma)\right).\end{aligned} \qquad \text{[A.1.12]}$$

Evaluate this derivative at $z = 1$:

$$\begin{aligned}\mathscr{F}''(1) = 2(1-\beta\gamma) + \frac{3}{\sigma}(1-\beta\gamma) + \frac{3}{\sigma}(1-\beta\gamma) - \frac{3\beta\gamma}{\sigma}\left(\frac{1}{\sigma}-1\right) + 2\beta(2+\gamma) \\ = 2(1+2\beta) + \frac{6}{\sigma}(1-\beta\gamma) - \frac{3\beta\gamma}{\sigma}\left(\frac{1}{\sigma}-1\right).\end{aligned} \qquad \text{[A.1.13]}$$

Note that when $\gamma/\sigma = (1+\beta)/\beta$:

$$\frac{1}{\sigma} = \frac{1+\beta}{\beta\gamma},$$

which can be substituted into [A.1.13] to obtain:

$$\begin{aligned}\mathscr{F}''(1) &= \frac{1}{\beta\gamma}\left(2\beta\gamma(1+2\beta) + 6(1+\beta)(1-\beta\gamma) - 3(1+\beta)((1+\beta)-\beta\gamma)\right) \\ &= \frac{1}{\beta\gamma}\left(2\beta\gamma(1+2\beta) + 3(1+\beta)(1-\beta-\beta\gamma)\right) = \frac{1-\beta}{\beta\gamma}\left(3(1+\beta)-\beta\gamma\right).\end{aligned} \qquad \text{[A.1.14]}$$

Therefore, from [A.1.10] and [A.1.14], when $\gamma/\sigma = (1+\beta)/\beta$ it follows that $\mathscr{F}'(1) = 0$ and $\mathscr{F}''(1) > 0$, the latter by noting $0 < \beta < 1$ and $0 < \gamma < 1$. Since $\mathscr{F}(1) = 0$, it must be the case that $\mathscr{F}(z)$ is positive in a neighbourhood below $z = 1$. Combined with [A.1.8], this means that there exists a solution of $\mathscr{F}(z) = 0$ in the economically meaningful range $(0, 1)$, demonstrating that the steady state $z = 1$ is not unique. Therefore, if [A.1.11] does not hold then there exist multiple steady states.

Sufficient conditions

Let $\mathcal{F}(z;\sigma)$ denote the function $\mathscr{F}(z)$ from [A.1.7] with the dependence on the parameter σ made explicit. This can be written as:

$$\mathcal{F}(z;\sigma) \equiv (1-\beta\gamma)(1+z+z^2)z^{\frac{1}{\sigma}} - 3z^{\frac{1}{\sigma}} + 3\beta z^2 - \beta(1-\gamma)(1+z+z^2). \quad \text{[A.1.15]}$$

First, consider the limiting case of $\sigma \to \infty$. For any $z > 0$, [A.1.15] reduces to:

$$\mathcal{F}(z;\infty) = (1-\beta\gamma)(1+z+z^2) - 3 + 3\beta z^2 - \beta(1-\gamma)(1+z+z^2) = (1+2\beta)z^2 + (1-\beta)z - (2+\beta).$$

This is a quadratic equation in z. Given that $0 < \beta < 1$, the coefficients of powers of z in the polynomial change sign exactly once. Hence, by Leibniz's rule of signs, the equation has at most one positive root. This shows that $z = 1$ is the unique positive solution of the equation $\mathcal{F}(z;\infty) = 0$. Furthermore, observe from [A.1.15] that $\mathcal{F}(0;\sigma) = -\beta(1-\gamma)$, so there cannot be a root at $z = 0$ even when $\sigma \to \infty$.

Next, consider the special case of $\sigma = 1$, in which case [A.1.15] reduces to:

$$\mathcal{F}(z;1) = (1-\beta\gamma)(1+z+z^2)z - 3z + 3\beta z^2 - \beta(1-\gamma)(1+z+z^2) = (1-\beta\gamma)z^3 + (1+2\beta)z^2 - (2+\beta)z - \beta(1-\gamma).$$

This is a cubic equation in z. Given that $0 < \beta < 1$ and $0 < \gamma < 1$, the coefficients of powers of z change sign exactly once. Hence, by Leibniz's rule of signs, the equation $\mathcal{F}(z;1) = 0$ can have no more than one positive root. This establishes that $z = 1$ is the unique positive solution of the equation.

Finally, consider one more special case, namely $\sigma = 1/2$. From [A.1.15]:

$$\begin{aligned}\mathcal{F}(z;1/2) &= (1-\beta\gamma)(1+z+z^2)z^2 - 3z^2 + 3\beta z^2 - \beta(1-\gamma)(1+z+z^2) \\ &= (1-\beta\gamma)z^4 + (1-\beta\gamma)z^3 - 2(1-\beta)z^2 - \beta(1-\gamma)z - \beta(1-\gamma).\end{aligned}$$

This is a quartic equation in z. Given that $0 < \beta < 1$ and $0 < \gamma < 1$, the coefficients of powers of z change sign exactly once. Hence, by Leibniz's rule of signs, the equation has at most one positive root, proving that $z = 1$ is the unique positive solution.

To analyse a general value of σ, define the function $\mathcal{H}(z)$ as follows:

$$\mathcal{H}(z) \equiv (1-\beta\gamma)(1+z+z^2) - 3. \quad \text{[A.1.16]}$$

Since $0 < \beta < 1$ and $0 < \gamma < 1$, the function $\mathcal{H}(z)$ is strictly increasing in z for all $z \geq 0$. Given that $\mathcal{H}(0) = -(2+\beta\gamma) < 0$, it follows $\mathcal{H}(z) = 0$ has a unique strictly positive root, denoted by $\bar{z}$. Since $\mathcal{H}(1) = -3\beta\gamma < 0$, it must be that $\bar{z} > 1$. The function $\mathscr{H}(z)$ is negative for $z < \bar{z}$ and positive for $z > \bar{z}$. Using [A.1.15], $\mathcal{F}(z;\sigma)$ can be written in terms of $\mathcal{H}(z)$ as follows:

$$\mathcal{F}(z;\sigma) = z^{\frac{1}{\sigma}}\mathcal{H}(z) + \beta\left(3z^2 - (1-\gamma)(1+z+z^2)\right). \quad \text{[A.1.17]}$$

Now use [A.1.17] to take the derivative of $\mathcal{F}(z;\sigma)$ with respect to σ, holding z constant:

$$\frac{\partial\mathcal{F}(z;\sigma)}{\partial\sigma} = -\frac{1}{\sigma^2}(\log z)z^{\frac{1}{\sigma}}\mathcal{H}(z). \quad \text{[A.1.18]}$$

If $z \in (0,1)$ then $\log z < 0$, while $\mathcal{H}(z) < 0$ since $\bar{z} > 1$. It follows that $\mathcal{F}(z;\sigma)$ is decreasing in σ in this range of z values. If $z \in (1,\bar{z})$ then $\mathcal{H}(z)$ is still negative, while $\log z > 0$, so $\mathcal{F}(z;\sigma)$ is strictly increasing in σ. Finally, if $z \in (\bar{z},\infty)$, $\mathcal{H}(z) > 0$ and $\log z > 0$ since $\bar{z} > 1$, so $\mathcal{F}(z;\sigma)$ is strictly decreasing in σ for these z values. It follows for any value of σ in the range $1/2 \leq \sigma \leq \infty$ that $\mathcal{F}(z;\sigma)$ lies somewhere between the values of $\mathcal{F}(z;1/2)$ and $\mathcal{F}(z;\infty)$. Formally, for all $0 \leq z \leq \infty$:

$$\min\{\mathcal{F}(z;1/2),\mathcal{F}(z;\infty)\} \leq \mathcal{F}(z;\sigma) \leq \max\{\mathcal{F}(z;1/2),\mathcal{F}(z;\infty)\}. \quad \text{[A.1.19]}$$

Now suppose the equation $\mathcal{F}(z;\sigma) = 0$ were to have a root $z \neq 1$. Given the bounds in [A.1.19] and the continuity of $\mathcal{F}(z;\sigma)$ for all σ, this is not possible unless either $\mathcal{F}(z;1/2)$ or $\mathcal{F}(z;\infty)$ also has a root $z \neq 1$, but not necessarily the same one as $\mathcal{F}(z;\sigma)$. As this has already been ruled out, it is therefore shown that $z=1$ is the only positive root of $\mathscr{F}(z)=0$ for any σ satisfying $1/2 \leq \sigma < \infty$.

(iii) Having determined the steady-state values of the real variables, the nominal variables must satisfy [A.1.1d]–[A.1.1e]. Substituting [A.1.1e] into [A.1.1d] and using the earlier result $\bar{r}=\varrho$:

$$(1+\bar{\pi})^{\psi_\pi - 1} = \frac{1+\varrho}{\psi_0}.$$

This has a unique solution for inflation whenever $\psi_\pi \neq 1$:

$$\bar{\pi} = \left(\frac{1+\varrho}{\psi_0}\right)^{\frac{1}{\psi_\pi - 1}} - 1,$$

which then determines the nominal interest rate using [A.1.1d]:

$$\bar{i} = (1+\varrho)(1+\bar{\pi}) - 1.$$

This completes the proof.

A.2 Lemmas

Lemma 1 *Let $\mathscr{G}(z)$ be the following quadratic equation:*

$$\mathscr{G}(z) \equiv \beta\left(1-\frac{\gamma}{\sigma}\right)z^2 + \left(2(1+\beta)-\frac{\beta\gamma}{\sigma}\right)z + \left(1-\frac{\beta\gamma}{\sigma}\right). \qquad \text{[A.2.1]}$$

Assume the parameters are such that $0<\beta<1$, $0<\gamma<1$, $\sigma>0$. If the following condition is satisfied:

$$\frac{\gamma}{\sigma} < \frac{1+\beta}{\beta}, \qquad \text{[A.2.2]}$$

then $\mathscr{G}(z)$ can be factorized uniquely as

$$\mathscr{G}(z) = \delta z(1-\lambda z^{-1})(1-\zeta z), \qquad \text{[A.2.3]}$$

in terms of the roots λ and ζ^{-1} of $\mathscr{G}(z)=0$ and a non-zero coefficient δ. These are given by the following formulas:

$$\lambda = \frac{\sqrt{(1+2\beta)^2 + 3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)} - (1+2\beta) - \left(1-\frac{\beta\gamma}{\sigma}\right)}{2\beta\left(1-\frac{\gamma}{\sigma}\right)}$$
$$= \frac{-2\left(1-\frac{\beta\gamma}{\sigma}\right)}{(1+2\beta)+\left(1-\frac{\beta\gamma}{\sigma}\right)+\sqrt{(1+2\beta)^2+3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}; \qquad \text{[A.2.4a]}$$

$$\zeta = \frac{-2\beta\left(1-\frac{\gamma}{\sigma}\right)}{(1+2\beta)+\left(1-\frac{\beta\gamma}{\sigma}\right)+\sqrt{(1+2\beta)^2+3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}; \qquad \text{[A.2.4b]}$$

$$\delta = \frac{1}{2}\left((1+2\beta)+\left(1-\frac{\beta\gamma}{\sigma}\right)+\sqrt{(1+2\beta)^2+3\left(1-\left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right). \qquad \text{[A.2.4c]}$$

The roots λ and ζ^{-1} are such that $|\lambda|<1$ and $|\zeta|<1$, and the coefficient δ satisfies $0<\delta<2(1+\beta)$.

PROOF Evaluate the quadratic $\mathscr{G}(z)$ in [A.2.1] at $z = -1$ and $z = 1$:

$$\mathscr{G}(-1) = -\left(1 + \beta\left(1 + \frac{\gamma}{\sigma}\right)\right) < 0, \quad \text{and } \mathscr{G}(1) = 3\left((1+\beta) - \frac{\beta\gamma}{\sigma}\right).$$

Given that condition [A.2.2] is assumed to hold, it follows that $\mathscr{G}(1) > 0$, and hence that $\mathscr{G}(z)$ changes sign over the interval $[-1, 1]$. Thus, by continuity, $\mathscr{G}(z) = 0$ always has a root in the interval $(-1, 1)$. Let this root be denoted by λ, which must satisfy $|\lambda| < 1$.

Since [A.2.1] holds, it must be the case that

$$2(1+\beta) > \frac{\beta\gamma}{\sigma},$$

and thus that the coefficient of z in [A.2.1] is never zero. The coefficient of z^2 can be zero, though, so $\mathscr{G}(z)$ is either quadratic or purely linear. This means that either $\mathscr{G}(z)$ has only one root or has two distinct roots. As one root is known to be real, complex roots are not possible. Given that $\mathscr{G}(z)$ is at most quadratic and has a sign change on $[-1, 1]$, there can be no more than one root in this interval. A second root, if it exists, lies in either $(-\infty, -1)$ or $(1, \infty)$. If there is a second root, let ζ denote the reciprocal of this root. If there is no second root, let $\zeta = 0$. In either case, ζ is a real number satisfying $|\zeta| < 1$.

When $\zeta = 0$, the function given in [A.2.3] is linear with single root at $z = \lambda$. When $\zeta \neq 0$, [A.2.3] is a quadratic function with roots $z = \lambda$ and $z = \zeta^{-1}$. Therefore, the factorization [A.2.3] must hold for some non-zero coefficient δ.

Take the case of $\gamma < \sigma$ first. This means the coefficient of z^2 in [A.2.1] is positive, so the quadratic is u-shaped. Given that $\mathscr{G}(-1) < 0$, it follows that the second root ζ^{-1} is found in $(-\infty, -1)$, that is, to the left of λ. Now consider the case of $\gamma > \sigma$, where the coefficient of z^2 in $\mathscr{G}(z)$ is negative, and $\mathscr{G}(z)$ is n-shaped. With $\mathscr{G}(1) > 0$ this means that the second root ζ^{-1} is found in $(1, \infty)$, lying to the right of λ. In applying the quadratic root formula to find λ, observe that the denominator of the formula is positive in the case where $\gamma < \sigma$ (with $\zeta^{-1} < \lambda$) and negative when $\gamma > \sigma$ (with $\lambda < \zeta^{-1}$). Therefore, the root λ is always associated with the upper branch of the quadratic root function:

$$\lambda = \frac{-\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right) + \sqrt{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right)^2 - 4\beta\left(1 - \frac{\gamma}{\sigma}\right)\left(1 - \frac{\beta\gamma}{\sigma}\right)}}{2\beta\left(1 - \frac{\gamma}{\sigma}\right)}. \qquad \text{[A.2.5]}$$

Since λ is known to be a real number, the term inside the square root must be non-negative.

When a second root exists, ζ^{-1} is given by the lower branch of the quadratic root function, and hence an expression for ζ is:

$$\zeta = \frac{-2\beta\left(1 - \frac{\gamma}{\sigma}\right)}{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right) + \sqrt{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right)^2 - 4\beta\left(1 - \frac{\gamma}{\sigma}\right)\left(1 - \frac{\beta\gamma}{\sigma}\right)}}. \qquad \text{[A.2.6]}$$

Using [A.2.1], the formula for the product $\lambda\zeta^{-1}$ of the roots of $\mathscr{G}(z) = 0$ implies:

$$\lambda = \frac{\left(1 - \frac{\beta\gamma}{\sigma}\right)}{\beta\left(1 - \frac{\gamma}{\sigma}\right)}\zeta. \qquad \text{[A.2.7]}$$

Substituting for ζ from [A.2.6] provides an alternative expression for λ:

$$\lambda = \frac{-2\left(1 - \frac{\beta\gamma}{\sigma}\right)}{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right) + \sqrt{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right)^2 - 4\beta\left(1 - \frac{\gamma}{\sigma}\right)\left(1 - \frac{\beta\gamma}{\sigma}\right)}}. \qquad \text{[A.2.8]}$$

Given that condition [A.2.2] holds and that the term in the square root is positive, [A.2.8] provides a well-defined formula for λ in all cases, including $\gamma = \sigma$. Similarly, it can be seen from [A.2.6] that $\zeta = 0$ if and only if $\gamma = \sigma$, which given the definition [A.2.1] is equivalent to $\mathscr{G}(z)$ being purely linear. Therefore, formulas [A.2.6] and [A.2.8] are well defined for all configurations of γ and σ consistent with [A.2.2].

Multiplying out the terms in the factorization [A.2.3] yields:

$$\mathscr{G}(z) = -\delta\zeta z^2 + \delta(1 + \lambda\zeta)z - \delta\lambda.$$

Equating the constant term with that in [A.2.1] implies $-\delta\lambda = (1 - \beta\gamma/\sigma)$, which leads to the following expression for δ:

$$\delta = \frac{-\left(1 - \frac{\beta\gamma}{\sigma}\right)}{\lambda}.$$

Substituting for λ from [A.2.8] shows that δ is given by:

$$\delta = \frac{1}{2}\left(\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right) + \sqrt{\left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right)^2 - 4\beta\left(1 - \frac{\gamma}{\sigma}\right)\left(1 - \frac{\beta\gamma}{\sigma}\right)}\right). \qquad \text{[A.2.9]}$$

Given [A.2.2] holds and the term in the square root is positive, it follows that δ is strictly positive. Observe that the term in the square root can be simplified as follows:

$$\left(2(1+\beta) - \beta\frac{\gamma}{\sigma}\right)^2 - 4\beta\left(1 - \frac{\gamma}{\sigma}\right)\left(1 - \frac{\beta\gamma}{\sigma}\right) = (1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right). \qquad \text{[A.2.10]}$$

It follows that the term inside the square root is never more than $(1+2\beta)^2 + 3$, and since $(1+2\beta)^2 + 3 \leq (2(1+\beta))^2$, it can be seen from [A.2.9] that $\delta \leq 2(1+\beta)$.

Finally, substituting [A.2.10] into [A.2.5], [A.2.8], [A.2.6], and [A.2.9] yield the formulas in [A.2.4a]–[A.2.4c]. This completes the proof. ■

A.3 Proof of Proposition 2

In what follows, it is helpful to define a new variable e_t in terms of r_t and g_t:

$$\mathsf{e}_t \equiv \mathsf{r}_t - \mathsf{g}_t. \qquad \text{[A.3.1]}$$

The system of equations [3.2a]–[3.2b] under incomplete markets can then be written in terms of the variables d_t, e_t, $\mathsf{c}_{\mathrm{y},t}$, $\mathsf{c}_{\mathrm{m},t}$, $\mathsf{c}_{\mathrm{o},t}$:

$$\mathsf{c}_{\mathrm{y},t} = \beta\gamma\mathsf{d}_t, \qquad \text{[A.3.2a]}$$

$$\mathsf{c}_{\mathrm{m},t} = -\beta\gamma\mathsf{d}_t - \gamma(\mathsf{e}_t + \mathsf{d}_{t-1}), \qquad \text{[A.3.2b]}$$

$$\mathsf{c}_{\mathrm{o},t} = \gamma(\mathsf{e}_t + \mathsf{d}_{t-1}), \qquad \text{[A.3.2c]}$$

$$\mathsf{c}_{\mathrm{y},t} = \mathbb{E}_t\mathsf{c}_{\mathrm{m},t+1} - \sigma\mathbb{E}_t\mathsf{e}_{t+1} + (1-\sigma)\mathbb{E}_t\mathsf{g}_{t+1}, \qquad \text{[A.3.2d]}$$

$$\mathsf{c}_{\mathrm{m},t} = \mathbb{E}_t\mathsf{c}_{\mathrm{o},t+1} - \sigma\mathbb{E}_t\mathsf{e}_{t+1} + (1-\sigma)\mathbb{E}_t\mathsf{g}_{t+1}. \qquad \text{[A.3.2e]}$$

This is a system of five equations in five unknowns if real GDP growth g_t is taken as given. However, given the presence of expectations of the future, this will not suffice to determine a unique solution in general. Nonetheless, it will be possible to characterize the set of possible equilibria for the debt ratio d_t up to a martingale difference.

(i) First, substitute the budget identities [A.3.2a] and [A.3.2b] for the young and middle-aged into the Euler equation [A.3.2d] for the young:

$$\beta\gamma\mathsf{d}_t = \mathbb{E}_t[-\beta\gamma\mathsf{d}_{t+1} - \gamma(\mathsf{e}_{t+1} + \mathsf{d}_t)] - \sigma\mathbb{E}_t\mathsf{e}_{t+1} + (1-\sigma)\mathbb{E}_t\mathsf{g}_{t+1}.$$

This can be rearranged to obtain an expression for $\mathbb{E}_t \mathsf{e}_{t+1}$:

$$(\gamma + \sigma)\mathbb{E}_t \mathsf{e}_{t+1} = -\gamma(1+\beta)\mathsf{d}_t - \beta\gamma\mathbb{E}_t \mathsf{d}_{t+1} + (1-\sigma)\mathbb{E}_t \mathsf{g}_{t+1}. \quad \text{[A.3.3]}$$

Next, combine the Euler equations [A.3.2d] and [A.3.2e] for both the young and the middle-aged:

$$\mathsf{c}_{\mathrm{y},t} = \mathbb{E}_t[\mathbb{E}_{t+1}\mathsf{c}_{\mathrm{o},t+2} - \sigma\mathbb{E}_{t+1}\mathsf{e}_{t+2} + (1-\sigma)\mathbb{E}_{t+1}\mathsf{g}_{t+2}] - \sigma\mathbb{E}_t\mathsf{e}_{t+1} + (1-\sigma)\mathbb{E}_t\mathsf{g}_{t+1},$$

which simplifies to:

$$\mathsf{c}_{\mathrm{y},t} = \mathbb{E}_t\mathsf{c}_{\mathrm{o},t+2} - \sigma\mathbb{E}_t[\mathsf{e}_{t+1} + \mathsf{e}_{t+2}] + (1-\sigma)\mathbb{E}_t[\mathsf{g}_{t+1} + \mathsf{g}_{t+2}].$$

Then substitute the budget identities [A.3.2a] and [A.3.2c] for the young and old into this equation:

$$\beta\gamma\mathsf{d}_t = \mathbb{E}_t[\gamma(\mathsf{e}_{t+2} + \mathsf{d}_{t+1})] - \sigma\mathbb{E}_t[\mathsf{e}_{t+1} + \mathsf{e}_{t+2}] + (1-\sigma)\mathbb{E}_t[\mathsf{g}_{t+1} + \mathsf{g}_{t+2}],$$

and rearrange this collect the terms in $\mathbb{E}_t\mathsf{e}_{t+1}$ and $\mathbb{E}_t\mathsf{e}_{t+2}$ on the left-hand side:

$$(\sigma - \gamma)\mathbb{E}_t\mathsf{e}_{t+2} + \sigma\mathbb{E}_t\mathsf{e}_{t+1} = -\beta\gamma\mathsf{d}_t + \gamma\mathbb{E}_t\mathsf{d}_{t+1} + (1-\sigma)\mathbb{E}_t[\mathsf{g}_{t+1} + \mathsf{g}_{t+2}]. \quad \text{[A.3.4]}$$

Multiply both sides of the equation [A.3.4] by the positive coefficient $(\gamma + \sigma)$ and apply the law of iterated expectations to write:

$$(\sigma-\gamma)\mathbb{E}_t\left[(\gamma+\sigma)\mathbb{E}_{t+1}\mathsf{e}_{t+2}\right] + \sigma\left((\gamma+\sigma)\mathbb{E}_t\mathsf{e}_{t+1}\right) = -\beta\gamma(\gamma+\sigma)\mathsf{d}_t + \gamma(\gamma+\sigma)\mathbb{E}_t\mathsf{d}_{t+1} + (1-\sigma)(\gamma+\sigma)\mathbb{E}_t[\mathsf{g}_{t+1} + \mathsf{g}_{t+2}].$$

Now substitute the expression for $(\gamma + \sigma)\mathbb{E}_t\mathsf{e}_{t+1}$ from [A.3.3] into the above equation:

$$(\sigma-\gamma)\mathbb{E}_t\left[-\gamma(1+\beta)\mathsf{d}_{t+1} - \beta\gamma\mathbb{E}_{t+1}\mathsf{d}_{t+2} + (1-\sigma)\mathbb{E}_{t+1}\mathsf{g}_{t+2}\right] + \sigma\left(-\gamma(1+\beta)\mathsf{d}_t - \beta\gamma\mathbb{E}_t\mathsf{d}_{t+1} + (1-\sigma)\mathbb{E}_t\mathsf{g}_{t+1}\right)$$
$$= -\beta\gamma(\gamma+\sigma)\mathsf{d}_t + \gamma(\gamma+\sigma)\mathbb{E}_t\mathsf{d}_{t+1} + (1-\sigma)(\gamma+\sigma)\mathbb{E}_t[\mathsf{g}_{t+1} + \mathsf{g}_{t+2}],$$

and by collecting terms:

$$\beta\gamma(\sigma-\gamma)\mathbb{E}_t\mathsf{d}_{t+2} + \gamma\left((1+\beta)(\sigma-\gamma) + \beta\sigma + (\gamma+\sigma)\right)\mathbb{E}_t\mathsf{d}_{t+1} + \gamma\left((1+\beta)\sigma - \beta(\gamma+\sigma)\right)\mathsf{d}_t$$
$$= (\sigma - (\gamma+\sigma))(1-\sigma)\mathbb{E}_t\mathsf{g}_{t+1} + ((\sigma-\gamma) - (\gamma+\sigma))(1-\sigma)\mathbb{E}_t\mathsf{g}_{t+2},$$

which can be simplified to yield the following equation:

$$\beta\gamma(\sigma-\gamma)\mathbb{E}_t\mathsf{d}_{t+2} + \gamma(2(1+\beta)\sigma - \beta\gamma)\mathbb{E}_t\mathsf{d}_{t+1} + \gamma(1-\beta\gamma)\mathsf{d}_t = -\gamma(1-\sigma)\mathbb{E}_t[\mathsf{g}_{t+1} + 2\mathsf{g}_{t+2}].$$

Finally, divide both sides by the positive coefficient $\gamma\sigma$:

$$\beta\left(1 - \frac{\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+2} + \left(2(1+\beta) - \frac{\beta\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+1} + \left(1 - \frac{\beta\gamma}{\sigma}\right)\mathsf{d}_t = -\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t[\mathsf{g}_{t+1} + 2\mathsf{g}_{t+2}]. \quad \text{[A.3.5]}$$

The dynamic equation [A.3.5] for the debt ratio d_t can be written in terms of the quadratic equation $\mathscr{G}(z)$ from [A.2.1] analysed in Lemma 1. If $\mathbb{L}$ denotes the lag operator, $\mathbb{F}$ the forward operator, and $\mathbb{I}$ the identity operator, equation [A.3.5] is:

$$\mathbb{E}_t\left[\mathscr{G}(\mathbb{F})\mathsf{d}_t\right] = -\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\left[(\mathbb{I} + 2\mathbb{F})\mathbb{F}\mathsf{g}_t\right].$$

As it has been assumed the parameters are such that there is a unique steady state, Proposition 1 implies that $\sigma > \gamma/(2+\varrho)$. With the definition of β in [2.1], this is seen to be equivalent to [A.2.2], and therefore the results of Lemma 1 can be applied. The factorization of $\mathscr{G}(z)$ from [A.2.3] can be used to obtain:

$$\mathbb{E}_t\left[\delta(\mathbb{I} - \zeta\mathbb{F})(\mathbb{I} - \lambda\mathbb{L})\mathbb{F}\mathsf{d}_t\right] = -\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\left[(\mathbb{I} + 2\mathbb{F})\mathbb{F}\mathsf{g}_t\right],$$

where λ, ζ, and δ are the terms from [A.2.4a]–[A.2.4c], and therefore:

$$(\mathbb{E}_t \mathsf{d}_{t+1} - \lambda \mathsf{d}_t) - \zeta \mathbb{E}_t \left[\mathbb{E}_{t+1} \mathsf{d}_{t+2} - \lambda \mathsf{d}_{t+1}\right] = -\frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right) \mathbb{E}_t \left[\mathsf{g}_{t+1} + 2\mathsf{g}_{t+2}\right]. \quad \text{[A.3.6]}$$

Now consider the stochastic process f_t defined in [3.4] using the coefficient ζ from [A.2.4b], which is the same as that defined in [3.5c]. Given a bounded stochastic process g_t for real GDP growth, and since Lemma 1 demonstrates that $|\zeta| < 1$, it follows that f_t is also a bounded stochastic process.

To study the general class of solutions of [A.3.2a]–[A.3.2e], using f_t from [3.4] and the coefficient λ from [A.2.4a], define Υ_t as follows:

$$\Upsilon_t \equiv \mathbb{E}_t \mathsf{d}_{t+1} - \lambda \mathsf{d}_t + \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(2\mathbb{E}_t \mathsf{f}_{t+1} + \mathsf{f}_t). \quad \text{[A.3.7]}$$

Given that g_t is exogenous, so is the stochastic process f_t. Since d_t is endogenous, so is the stochastic process Υ_t.

Observe that the definition of f_t in [3.4] implies that it satisfies the following recursive equation:

$$\mathsf{f}_t - \zeta \mathbb{E}_t \mathsf{f}_{t+1} = \mathbb{E}_t \mathsf{g}_{t+1}. \quad \text{[A.3.8]}$$

Next, from the definition of Υ_t in [A.3.7]:

$$\mathbb{E}_t \mathsf{d}_{t+1} = \lambda \mathsf{d}_t - \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(2\mathbb{E}_t \mathsf{f}_{t+1} + \mathsf{f}_t) + \Upsilon_t, \quad \text{[A.3.9]}$$

and by substituting this into [A.3.6]:

$$\Upsilon_t - \zeta \mathbb{E}_t \Upsilon_{t+1} - \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right) \mathbb{E}_t \left[2(\mathsf{f}_{t+1} - \zeta \mathbb{E}_{t+1} \mathsf{f}_{t+2}) + (\mathsf{f}_t - \zeta \mathbb{E}_t \mathsf{f}_{t+1})\right] = -\frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right) \mathbb{E}_t \left[\mathsf{g}_{t+1} + 2\mathsf{g}_{t+2}\right].$$

Using [A.3.8], this equation reduces to:

$$\Upsilon_t = \zeta \mathbb{E}_t \Upsilon_{t+1}, \quad \text{[A.3.10]}$$

which characterizes the whole class of solutions for Υ_t, and through [A.3.7], the class of solutions for the debt ratio d_t.

Now consider the remaining Euler equation and budget identity combination. Take the budget identities [A.3.2b] and [A.3.2c] of the middle-aged and old and substitute these into the Euler equation [A.3.2e] for the middle-aged:

$$(-\beta\gamma \mathsf{d}_t - \gamma(\mathsf{e}_t + \mathsf{d}_{t-1})) = \mathbb{E}_t \left[\gamma(\mathsf{e}_{t+1} + \mathsf{d}_t)\right] - \sigma \mathbb{E}_t \mathsf{e}_{t+1} + (1-\sigma)\mathbb{E}_t \mathsf{g}_{t+1}.$$

Collect the terms in $\mathbb{E}_t \mathsf{e}_{t+1}$ on the left-hand side to obtain:

$$(\sigma - \gamma)\mathbb{E}_t \mathsf{e}_{t+1} = \gamma(1+\beta)\mathsf{d}_t + \gamma(\mathsf{e}_t + \mathsf{d}_{t-1}) + (1-\sigma)\mathbb{E}_t \mathsf{g}_{t+1},$$

then multiplying both sides by the positive coefficient $(\gamma + \sigma)$ and applying the law of iterated expectations:

$$(\sigma - \gamma)\left((\gamma + \sigma)\mathbb{E}_t \mathsf{e}_{t+1}\right) = \gamma(1+\beta)(\gamma+\sigma)\mathsf{d}_t + \gamma(\gamma+\sigma)(\mathsf{e}_t + \mathsf{d}_{t-1}) + (1-\sigma)(\gamma+\sigma)\mathbb{E}_t \mathsf{g}_{t+1}.$$

Substitute the expression for $(\gamma + \sigma)\mathbb{E}_t \mathsf{e}_{t+1}$ from [A.3.3]:

$$\begin{aligned} &(\sigma - \gamma)\left(-\gamma(1+\beta)\mathsf{d}_t - \beta\gamma \mathbb{E}_t \mathsf{d}_{t+1} + (1-\sigma)\mathbb{E}_t \mathsf{g}_{t+1}\right) \\ &= \gamma(1+\beta)(\gamma+\sigma)\mathsf{d}_t + \gamma(\gamma+\sigma)(\mathsf{e}_t + \mathsf{d}_{t-1}) + (1-\sigma)(\gamma+\sigma)\mathbb{E}_t \mathsf{g}_{t+1}, \end{aligned}$$

then collecting similar terms:

$$\gamma(\gamma+\sigma)(\mathsf{e}_t + \mathsf{d}_{t-1}) + \gamma(1+\beta)\left((\gamma+\sigma) + (\sigma-\gamma)\right)\mathsf{d}_t + \beta\gamma(\sigma-\gamma)\mathbb{E}_t \mathsf{d}_{t+1} = (1-\sigma)((\sigma-\gamma) - (\gamma+\sigma))\mathbb{E}_t \mathsf{g}_{t+1},$$

and rearranging to obtain:

$$\gamma(\gamma+\sigma)(\mathsf{e}_t+\mathsf{d}_{t-1})+2\gamma\sigma(1+\beta)\mathsf{d}_t+\beta\gamma(\sigma-\gamma)\mathbb{E}_t\mathsf{d}_{t+1}=-2\gamma(1-\sigma)\mathbb{E}_t\mathsf{g}_{t+1}.$$

Finally, dividing both sides by the non-zero coefficient $\gamma\sigma$:

$$\left(1+\frac{\gamma}{\sigma}\right)(\mathsf{e}_t+\mathsf{d}_{t-1})+2(1+\beta)\mathsf{d}_t+\beta\left(1-\frac{\gamma}{\sigma}\right)\mathbb{E}_t\mathsf{d}_{t+1}=-2\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\mathsf{g}_{t+1}. \quad \text{[A.3.11]}$$

A general solution to the debt ratio equation [A.3.5] has been shown to satisfy [A.3.9], in terms of a stochastic process Υ_t which in turn satisfies the expectational difference equation [A.3.10]. Substituting [A.3.9] into [A.3.11]:

$$\left(1+\frac{\gamma}{\sigma}\right)(\mathsf{e}_t+\mathsf{d}_{t-1})+2(1+\beta)\mathsf{d}_t+\beta\left(1-\frac{\gamma}{\sigma}\right)\left(\lambda\mathsf{d}_t-\frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(2\mathbb{E}_t\mathsf{f}_{t+1}+\mathsf{f}_t)+\Upsilon_t\right)=-2\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\mathsf{g}_{t+1},$$

which yields the following after collecting terms:

$$\begin{aligned}\left(1+\frac{\gamma}{\sigma}\right)(\mathsf{e}_t+\mathsf{d}_{t-1})+\left(2(1+\beta)+\beta\left(1-\frac{\gamma}{\sigma}\right)\lambda\right)\mathsf{d}_t+\beta\left(1-\frac{\gamma}{\sigma}\right)\Upsilon_t\\ =-\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\left[2\mathsf{g}_{t+1}-\frac{\beta}{\delta}\left(1-\frac{\gamma}{\sigma}\right)(2\mathsf{f}_{t+1}+\mathsf{f}_t)\right].\end{aligned} \quad \text{[A.3.12]}$$

The formulas [A.2.4b] and [A.2.4c] for ζ and δ imply that $\zeta=\beta(1-\gamma/\sigma)/\delta$. Using this result and the expression for g_t in [A.3.8] to simplify the expectation on the right-hand side:

$$\mathbb{E}_t\left[2\mathsf{g}_{t+1}-\frac{\beta}{\delta}\left(1-\frac{\gamma}{\sigma}\right)(\mathsf{f}_{t+1}+2\mathsf{f}_{t+2})\right]=\mathbb{E}_t\left[2(\mathsf{f}_{t+1}-\zeta\mathsf{f}_{t+2})+\zeta(\mathsf{f}_{t+1}+2\mathsf{f}_{t+2})\right]=(2+\zeta)\mathbb{E}_t\mathsf{f}_{t+1}.$$

Substituting this back into [A.3.12], and noting that [A.2.4b] and [A.2.4c] imply $\beta(1-\gamma/\sigma)=\delta\zeta$, yields the following equation:

$$\left(1+\frac{\gamma}{\sigma}\right)(\mathsf{e}_t+\mathsf{d}_{t-1})+\left(2(1+\beta)+\beta\left(1-\frac{\gamma}{\sigma}\right)\lambda\right)\mathsf{d}_t=\delta\zeta\Upsilon_t-(2+\zeta)\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\mathsf{f}_{t+1}. \quad \text{[A.3.13]}$$

Define the following coefficients (can show that all are strictly positive):

$$\theta\equiv\frac{2(1+\beta)+\beta\left(1-\frac{\gamma}{\sigma}\right)\lambda}{1+\frac{\gamma}{\sigma}},\quad \vartheta\equiv\frac{2+\zeta}{1+\frac{\gamma}{\sigma}},\quad \text{and}\quad \varkappa\equiv\frac{\delta}{1+\frac{\gamma}{\sigma}}, \quad \text{[A.3.14]}$$

using which equation [A.3.13] can be written as:

$$(\mathsf{e}_t+\mathsf{d}_{t-1})+\theta\mathsf{d}_t=\zeta\varkappa\Upsilon_t-\vartheta\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\mathsf{f}_{t+1}. \quad \text{[A.3.15]}$$

To derive the class of solutions to the debt equation, define $\mathsf{l}_t\equiv\mathsf{e}_t+\mathsf{d}_{t-1}$ to be the total stock of liabilities relative to GDP. The generational budget identities [A.3.2a]–[A.3.2c] can be written as:

$$\mathsf{c}_{\mathrm{y},t}=\beta\gamma\mathsf{d}_t,\quad \mathsf{c}_{\mathrm{m},t}=-\beta\gamma\mathsf{d}_t-\gamma\mathsf{l}_t,\quad \text{and}\quad \mathsf{c}_{\mathrm{o},t}=\gamma\mathsf{l}_t, \quad \text{[A.3.16]}$$

and l_t is determined by equation [A.3.15]:

$$\mathsf{l}_t+\theta\mathsf{d}_t=\zeta\varkappa\Upsilon_t-\vartheta\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\mathsf{f}_{t+1}. \quad \text{[A.3.17]}$$

Now define ν_t to be the unpredictable component of the debt ratio d_t, and υ_t to be the unpredictable component of Υ_t:

$$\nu_t \equiv \mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t, \quad \text{and} \quad \upsilon_t \equiv \Upsilon_t - \mathbb{E}_{t-1}\Upsilon_t. \qquad [A.3.18]$$

First, consider the special case where $\zeta = 0$. Equation [A.3.10] then implies $\Upsilon_t = 0$. Next, consider the general case of $\zeta \neq 0$. Using [A.3.18], equation [A.3.10] can be expressed equivalently as

$$\Upsilon_t = \zeta^{-1}\Upsilon_{t-1} + \upsilon_t. \qquad [A.3.19]$$

Similarly, using [A.3.18], equation [A.3.9] is equivalent to:

$$\mathsf{d}_{t+1} = \lambda \mathsf{d}_t - \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t[\mathsf{f}_{t+1} + 2\mathsf{f}_{t+2}] + \Upsilon_t + \nu_{t+1}. \qquad [A.3.20]$$

Since $|\zeta| < 1$ and because υ_t must be a martingale difference sequence ($\mathbb{E}_{t-1}\upsilon_t = 0$) it is clear that $\Upsilon_t = 0$ is the only bounded solution of the equation [A.3.19]. Furthermore, since $\mathbb{E}_t\nu_{t+1} = 0$, the innovation ν_{t+1} must be uncorrelated with Υ_t. It follows from [A.3.20] that whenever Υ_t is unbounded (whenever $\Upsilon_t \neq 0$), the solution for d_t is also unbounded. Therefore, given [A.3.16] if either d_t or l_t were unbounded then the implied path for one of $\mathsf{c}_{\mathrm{y},t}$, $\mathsf{c}_{\mathrm{m},t}$, or $\mathsf{c}_{\mathrm{o},t}$ would be such as to violate one of the non-negativity constraints on consumption. This cannot be a solution, from which it follows that Υ_t cannot be unbounded. Finally, it then follows that $\Upsilon_t = 0$, which requires $\upsilon_t = 0$.

Unlike υ_t, there is no tight restriction that can be placed on ν_t. The only requirement imposed by equations [A.3.2a]–[A.3.2e] is that ν_t is a bounded martingale difference process.

Therefore, from [A.3.20] it is established that all solutions for the debt ratio d_t are in the following class

$$\mathsf{d}_{t+1} = \lambda \mathsf{d}_t - \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t[\mathsf{f}_{t+1} + 2\mathsf{f}_{t+2}] + \nu_{t+1}, \qquad [A.3.21]$$

where the bounded stochastic process f_t is as defined in [3.4], and ν_t is bounded and satisfies $\mathbb{E}_t\nu_{t+1} = 0$.

(ii) Given a solution for d_t, equation [A.3.15] gives the solution for e_t and hence r_t:

$$\mathsf{e}_t = -\theta \mathsf{d}_t - \mathsf{d}_{t-1} - \vartheta\left(\frac{1-\sigma}{\sigma}\right)\mathbb{E}_t\mathsf{f}_{t+1}. \qquad [A.3.22]$$

This completes the proof.

A.4 Proof of Proposition 3

The system of equations describing the economy with complete markets is [3.10a]–[3.10c].

(i) Note that equations [3.10a]–[3.10b] are of exactly the same form as [3.2a]–[3.2b] for the economy with incomplete markets. Proposition 2 can then be applied to deduce that the debt-GDP ratio d^*_t must satisfy equation [3.3], that is:

$$\mathsf{d}^*_t = \lambda \mathsf{d}^*_{t-1} - \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(2\mathbb{E}_{t-1}\mathsf{f}_t + \mathsf{f}_{t-1}) + \nu^*_t, \qquad [A.4.1]$$

where ν^*_t is such that $\mathbb{E}_{t-1}\nu^*_t = 0$, and where f_t is as defined in [3.4] and the coefficients δ and λ in [3.5a] and [3.5b]. Moreover, equation [3.6] must hold. Written in terms of $\mathsf{e}^*_t \equiv \mathsf{r}^*_t - \mathsf{g}_t$ this becomes:

$$\mathsf{e}^*_t = -\theta \mathsf{d}^*_t - \mathsf{d}^*_{t-1} - \vartheta\left(\frac{1-\sigma}{\sigma}\right)\mathsf{f}_t, \qquad [A.4.2]$$

where the coefficients θ and ϑ are given in [3.7a]–[3.7b].

Equation [A.4.1] implies $\mathsf{d}^*_t - \mathbb{E}_{t-1}\mathsf{d}^*_t = \nu^*_t$. Taking the unpredictable components of both sides of [A.4.2] and using the equation for ν^*_t yields:

$$(\mathsf{e}^*_t - \mathbb{E}_{t-1}\mathsf{e}^*_t) = -\theta\nu^*_t - \vartheta\left(\frac{1-\sigma}{\sigma}\right)(\mathsf{f}_t - \mathbb{E}_{t-1}\mathsf{f}_t). \qquad [A.4.3]$$

Now take the unpredictable components of both sides of equation [3.10c], which must hold under complete markets:

$$c^*_{m,t} - \mathbb{E}_{t-1}c^*_{m,t} = c^*_{o,t} - \mathbb{E}_{t-1}c^*_{o,t}. \quad \text{[A.4.4]}$$

The budget constraints in [3.10a] imply that $c^*_{m,t} = -\beta\gamma d^*_t - \gamma(e^*_t + d^*_{t-1})$ and $c^*_{o,t} = \gamma(e^*_t + d^*_{t-1})$. Substituting these into [A.4.4] and using the definition of ν^*_t:

$$-\beta\gamma\nu^*_t - \gamma(e^*_t - \mathbb{E}_{t-1}e^*_t) = \gamma(e^*_t - \mathbb{E}_{t-1}e^*_{t-1}),$$

and by collecting terms and cancelling the positive coefficient γ:

$$2(e^*_t - \mathbb{E}_{t-1}e^*_t) = -\beta\nu^*_t. \quad \text{[A.4.5]}$$

Multiplying both sides of [A.4.3] by 2 and substituting using [A.4.5]:

$$(2\theta - \beta)\nu^*_t = -2\vartheta\left(\frac{1-\sigma}{\sigma}\right)(f_t - \mathbb{E}_{t-1}f_t). \quad \text{[A.4.6]}$$

Using the expression for θ from [3.7a]:

$$2\theta - \beta = \frac{2 + \beta + \sqrt{(1+2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}}{1 + \frac{\gamma}{\sigma}},$$

and hence from [3.5a] and [3.7b] that δ, θ, and ϑ are related by:

$$\vartheta = \frac{2\theta - \beta}{\delta}. \quad \text{[A.4.7]}$$

Substituting this into [A.4.6] yields:

$$\nu^*_t = -\frac{2}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(f_t - \mathbb{E}_{t-1}f_t). \quad \text{[A.4.8]}$$

Finally, combining [A.4.1] and [A.4.8] leads to [3.11].

Next, take conditional expectations of [A.4.1] to obtain:

$$\mathbb{E}_{t-1}d^*_t = \lambda d^*_{t-1} - \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(2\mathbb{E}_{t-1}f_t + f_{t-1}).$$

Subtracting this from [3.8] and using the definition $\tilde{d}_t \equiv d_t - d^*_t$ implies that [3.12] holds.

(ii) Next, use equation [A.4.2] to deduce:

$$e^*_t - \lambda e^*_{t-1} = -\theta(d^*_t - \lambda d^*_{t-1}) - (d^*_{t-1} - \lambda d^*_{t-2}) - \vartheta\left(\frac{1-\sigma}{\sigma}\right)(f_t - \lambda f_{t-1}),$$

and by substituting from [3.11]:

$$e^*_t - \lambda e^*_{t-1} = \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)(\theta(2f_t + f_{t-1}) + (2f_{t-1} + f_{t-2}) - \delta\vartheta(f_t - \lambda f_{t-1})).$$

After collecting similar terms, this can be written as:

$$e^*_t - \lambda e^*_{t-1} = \frac{1}{\delta}\left(\frac{1-\sigma}{\sigma}\right)((2\theta - \delta\vartheta)f_t + (2 + \theta + \delta\vartheta\lambda)f_{t-1} + f_{t-2}). \quad \text{[A.4.9]}$$

Since equation [A.4.7] implies $2\theta - \beta = \delta\vartheta$:

$$2\theta - \delta\vartheta = \beta. \tag{A.4.10}$$

Now observe from the formulas [3.5a], [3.5c], [3.7a], and [3.7b] that the following relationships hold between the coefficients:

$$\theta = \frac{\delta + \frac{\beta\gamma}{\sigma}}{1 + \frac{\gamma}{\sigma}}, \quad \text{and} \quad \vartheta = \frac{2 + \zeta}{1 + \frac{\gamma}{\sigma}}.$$

These expressions can be used to note:

$$2 + \theta + \delta\vartheta\lambda = \frac{2\left(1 + \frac{\gamma}{\sigma}\right) + \left(\delta + \frac{\beta\gamma}{\sigma}\right) + (\delta\lambda)(2 + \zeta)}{1 + \frac{\gamma}{\sigma}},$$

and since [3.5a] and [3.5b] imply $\delta\lambda = -(1 - \beta\gamma/\sigma)$:

$$2 + \theta + \delta\vartheta\lambda = \frac{2\frac{\gamma}{\sigma} + 3\frac{\beta\gamma}{\sigma} + \delta - \left(1 - \frac{\beta\gamma}{\sigma}\right)\zeta}{1 + \frac{\gamma}{\sigma}}. \tag{A.4.11}$$

By using the expressions for λ and ζ from [A.2.4a] and [A.2.4b], it follows that $(1 - \beta\gamma/\sigma)\zeta = \beta(1 - \gamma/\sigma)\lambda$. Substituting this into equation [A.4.11]:

$$2 + \theta + \delta\vartheta\lambda = \frac{2\frac{\gamma}{\sigma} + 3\frac{\beta\gamma}{\sigma} + \delta - \beta\left(1 - \frac{\gamma}{\sigma}\right)\lambda}{1 + \frac{\gamma}{\sigma}}. \tag{A.4.12}$$

Using the expression for δ from [A.2.4c] and the alternative formula for λ from [A.2.4a]:

$$\delta - \beta\left(1 - \frac{\gamma}{\sigma}\right)\lambda = \frac{1}{2}\left((1 + 2\beta) + \left(1 - \frac{\beta\gamma}{\sigma}\right) + \sqrt{(1 + 2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)}\right)$$
$$-\frac{1}{2}\left(\sqrt{(1 + 2\beta)^2 + 3\left(1 - \left(\frac{\beta\gamma}{\sigma}\right)^2\right)} - (1 + 2\beta) - \left(1 - \frac{\beta\gamma}{\sigma}\right)\right) = 2(1 + \beta) - \frac{\beta\gamma}{\sigma}.$$

Substituting this into [A.4.12] yields:

$$2 + \theta + \delta\vartheta\lambda = \frac{2\left(1 + \beta + \frac{\gamma}{\sigma} + \frac{\beta\gamma}{\sigma}\right)}{1 + \frac{\gamma}{\sigma}} = 2(1 + \beta). \tag{A.4.13}$$

Hence [A.4.9] together with equations [A.4.10] and [A.4.13] implies:

$$\mathsf{e}_t^* - \lambda\mathsf{e}_{t-1}^* = \frac{1}{\delta}\left(\frac{1 - \sigma}{\sigma}\right)(\beta\mathsf{f}_t + 2(1 + \beta)\mathsf{f}_{t-1} + \mathsf{f}_{t-2}).$$

Since $\mathsf{e}_t^* - \lambda\mathsf{e}_{t-1}^* = (\mathsf{r}_t^* - \lambda\mathsf{r}_{t-1}^*) - (\mathsf{g}_t - \lambda\mathsf{g}_{t-1}$, this demonstrates that equation [3.13] holds.

Using the definition of e_t^*, equation [A.4.2] implies:

$$\mathsf{r}_t^* = -\theta\mathsf{d}_t^* - \mathsf{d}_{t-1}^* + \mathsf{g}_t - \vartheta\left(\frac{1 - \sigma}{\sigma}\right)\mathbb{E}_t\mathsf{f}_{t+1}. \tag{A.4.14}$$

With the definitions $\tilde{\mathsf{r}}_t \equiv \mathsf{r}_t - \mathsf{r}_t^*$ and $\tilde{\mathsf{d}}_t \equiv \mathsf{d}_t - \mathsf{d}_t^*$, subtracting [A.4.14] from [3.6] yields [3.14].

(iii) Subtracting the generational budget constraints with incomplete markets [3.2a] from the budget constraints with complete markets [3.10a]:

$$\tilde{c}_{y,t} = \beta\gamma\tilde{d}_t, \quad \tilde{c}_{m,t} = -\beta\gamma\tilde{d}_t - \gamma(\tilde{r}_t + \tilde{d}_{t-1}), \quad \text{and} \ \ \tilde{c}_{o,t} = \gamma(\tilde{r}_t + \tilde{d}_{t-1}).$$

Noting that [3.14] implies $\tilde{r}_t + \tilde{d}_{t-1} = \theta\tilde{d}_t$ and substituting into the equations above yields:

$$\tilde{c}_{y,t} = \beta\gamma\tilde{d}_t, \quad \tilde{c}_{m,t} = \gamma(\theta - \beta)\tilde{d}_t, \quad \text{and} \ \ \tilde{c}_{o,t} = -\gamma\theta\tilde{d}_t.$$

Since $\tilde{C}_{i,t} = C_{i,t} - C^*_{i,t} = c_{i,t} - c^*_{i,t} = \tilde{c}_{i,t}$, the results in [3.15] follow.

(iv) With $\sigma = 1$, it follows immediately from [3.12] that $d^*_t = 0$. If $\mathbb{E}_t g_{t+1} = 0$ then $\mathbb{E}_t g_{t+\ell} = 0$ for all $\ell \geq 1$ by the law of iterated expectations. The definition of f_t in [3.4] then implies $f_t = 0$, and hence $d^*_t = 0$ by equation [3.12]. With either $\sigma = 1$ or $f_t = 0$, [3.14] implies that $r^*_t = g_t$. This completes the proof.

A.5 Proof of Proposition 9

The social welfare function:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0}\left[\frac{1}{3}\sum_{t=t_0-2}^{\infty}\beta^{t-t_0}\hat{\omega}^*_t\left\{\frac{C_{y,t}^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}} + \beta\frac{C_{m,t+1}^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}} + \beta^2\frac{C_{o,t+2}^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}}\right\}\right].$$

Since the weights $\hat{\omega}^*_t$ are calculated for the case where GDP is $\hat{Y}^*_t$, and since $\hat{Y}^*_t = A_t$, it follows that the weights are independent of policy and thus

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0}\left[\frac{1}{3}\sum_{t=t_0-2}^{\infty}\beta^{t-t_0}\frac{\hat{\aleph}^*_t}{\hat{c}^{*\,-\frac{1}{\sigma}}_{y,t}}\left\{\frac{C_{y,t}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \beta\frac{C_{m,t+1}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \beta^2\frac{C_{o,t+2}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}\right\}\right] + \mathscr{I},$$

where $\mathscr{I}$ denotes terms independent of policy, and where the weights have been replaced by the formula in (??). Changing the order of summation and noting that terms dated prior to t_0 are independent of policy because predetermined:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0}\left[\frac{1}{3}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left\{\frac{\hat{\aleph}^*_t}{\hat{c}^{*\,-\frac{1}{\sigma}}_{y,t}}\frac{C_{y,t}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \frac{\hat{\aleph}^*_{t-1}}{\hat{c}^{*\,-\frac{1}{\sigma}}_{y,t-1}}\frac{C_{m,t}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \frac{\hat{\aleph}^*_{t-2}}{\hat{c}^{*\,-\frac{1}{\sigma}}_{y,t-2}}\frac{C_{o,t}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}\right\}\right] + \mathscr{I}.$$

Note that the Lagrangian multipliers $\hat{\aleph}^*_t$ are such that:

$$\hat{\aleph}^*_{t-1} = \hat{\aleph}^*_t\left(\frac{\hat{c}^*_{y,t-1}}{\hat{c}^*_{m,t}}\right)^{-\frac{1}{\sigma}}, \quad \text{and} \ \ \hat{\aleph}^*_{t-2} = \hat{\aleph}^*_t\left(\frac{\hat{c}^*_{y,t-2}}{\hat{c}^*_{o,t}}\right)^{-\frac{1}{\sigma}}.$$

By substituting these into the formula for the social welfare function:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0}\left[\frac{1}{3}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\hat{\aleph}^*_t\left\{\frac{C_{y,t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c}^{*\,-\frac{1}{\sigma}}_{y,t}} + \frac{C_{m,t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c}^{*\,-\frac{1}{\sigma}}_{m,t}} + \frac{C_{o,t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c}^{*\,-\frac{1}{\sigma}}_{o,t}}\right\}\right] + \mathscr{I}.$$

This can be written in terms of consumption to GDP ratios:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0}\left[\frac{1}{3}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\hat{\aleph}^*_t Y_t^{1-\frac{1}{\sigma}}\left\{\frac{c_{y,t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c}^{*\,-\frac{1}{\sigma}}_{y,t}} + \frac{c_{m,t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c}^{*\,-\frac{1}{\sigma}}_{m,t}} + \frac{c_{o,t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c}^{*\,-\frac{1}{\sigma}}_{o,t}}\right\}\right] + \mathscr{I}.$$

Finally, using the fact that $Y_t/\hat{Y}_t = 1/\Delta_t$:

$$\mathscr{W}_{t_0} = \mathbb{E}_{t_0}\left[\frac{1}{3}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\hat{\aleph}_t^*\hat{Y}_t^{1-\frac{1}{\sigma}}\Delta_t^{-\left(1-\frac{1}{\sigma}\right)}\left\{\frac{c_{\mathrm{y},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{y},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{m},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{m},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{o},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{o},t}^{-\frac{1}{\sigma}}}\right\}\right]+\mathscr{I}.$$

First, observe that

$$c_{i,t}^{1-\frac{1}{\sigma}} = 1+\left(1-\frac{1}{\sigma}\right)\mathsf{c}_{i,t}+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)^2\mathsf{c}_{i,t}^2+\mathscr{O}_3,$$

where $\mathscr{O}_3$ denotes terms of order three and higher in the exogenous shocks. Similarly:

$$\frac{1}{\hat{c^*}_{i,t}^{-\frac{1}{\sigma}}} = 1+\frac{1}{\sigma}\hat{\mathsf{c}^*}_{i,t}+\frac{1}{2}\frac{1}{\sigma^2}\hat{\mathsf{c}^*}_{i,t}^2+\mathscr{O}_3.$$

Putting these results together implies that

$$\frac{c_{i,t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{i,t}^{-\frac{1}{\sigma}}} = \frac{1}{1-\frac{1}{\sigma}}+\mathsf{c}_{i,t}+\frac{\frac{1}{\sigma}}{1-\frac{1}{\sigma}}\hat{\mathsf{c}^*}_{i,t}+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)\mathsf{c}_{i,t}^2+\frac{1}{2}\frac{\frac{1}{\sigma^2}}{1-\frac{1}{\sigma}}\hat{\mathsf{c}^*}_{i,t}^2+\frac{1}{\sigma}\hat{\mathsf{c}^*}_{i,t}\mathsf{c}_{i,t}+\mathscr{O}_3.$$

Summing over all generations:

$$\begin{aligned}\frac{c_{\mathrm{y},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{y},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{m},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{m},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{o},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{o},t}^{-\frac{1}{\sigma}}} &= \frac{3}{1-\frac{1}{\sigma}}+\frac{\frac{1}{\sigma}}{1-\frac{1}{\sigma}}\left(\hat{\mathsf{c}^*}_{\mathrm{y},t}+\hat{\mathsf{c}^*}_{\mathrm{m},t}+\hat{\mathsf{c}^*}_{\mathrm{o},t}\right)\\ +\left(\mathsf{c}_{\mathrm{y},t}+\mathsf{c}_{\mathrm{m},t}+\mathsf{c}_{\mathrm{o},t}\right)+\frac{1}{2}\left(1-\frac{1}{\sigma}\right)\left(\mathsf{c}_{\mathrm{y},t}^2+\mathsf{c}_{\mathrm{m},t}^2+\mathsf{c}_{\mathrm{o},t}^2\right)&+\frac{1}{2}\frac{\frac{1}{\sigma^2}}{1-\frac{1}{\sigma}}\left(\hat{\mathsf{c}^*}_{\mathrm{y},t}^2+\hat{\mathsf{c}^*}_{\mathrm{m},t}^2+\hat{\mathsf{c}^*}_{\mathrm{o},t}^2\right)\\ &+\frac{1}{\sigma}\left(\hat{\mathsf{c}^*}_{\mathrm{y},t}\mathsf{c}_{\mathrm{y},t}+\hat{\mathsf{c}^*}_{\mathrm{m},t}\mathsf{c}_{\mathrm{m},t}+\hat{\mathsf{c}^*}_{\mathrm{o},t}\mathsf{c}_{\mathrm{o},t}\right)+\mathscr{O}_3.\end{aligned}$$

Since $(1/3)(c_{\mathrm{y},t}+c_{\mathrm{m},t}+c_{\mathrm{o},t}) = 1$ and $(1/3)(\hat{c^*}_{\mathrm{y},t}+\hat{c^*}_{\mathrm{m},t}+\hat{c^*}_{\mathrm{o},t}) = 1$, it follows that:

$$\left(\mathsf{c}_{\mathrm{y},t}+\mathsf{c}_{\mathrm{y},t}+\mathsf{c}_{\mathrm{y},t}\right) = -\frac{1}{2}\left(\mathsf{c}_{\mathrm{y},t}^2+\mathsf{c}_{\mathrm{y},t}^2+\mathsf{c}_{\mathrm{y},t}^2\right)+\mathscr{O}_3,\quad \left(\hat{\mathsf{c}^*}_{\mathrm{y},t}+\hat{\mathsf{c}^*}_{\mathrm{m},t}+\hat{\mathsf{c}^*}_{\mathrm{o},t}\right) = -\frac{1}{2}\left(\hat{\mathsf{c}^*}_{\mathrm{y},t}^2+\hat{\mathsf{c}^*}_{\mathrm{m},t}^2+\hat{\mathsf{c}^*}_{\mathrm{o},t}^2\right).$$

Substituting this into the earlier expression:

$$\begin{aligned}\frac{c_{\mathrm{y},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{y},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{m},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{m},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{o},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{o},t}^{-\frac{1}{\sigma}}} &= \frac{3}{1-\frac{1}{\sigma}}-\frac{1}{2}\frac{1}{\sigma}\left(\mathsf{c}_{\mathrm{y},t}^2+\mathsf{c}_{\mathrm{m},t}^2+\mathsf{c}_{\mathrm{o},t}^2\right)\\ -\frac{1}{2}\frac{1}{\sigma}\left(\hat{\mathsf{c}^*}_{\mathrm{y},t}^2+\hat{\mathsf{c}^*}_{\mathrm{m},t}^2+\hat{\mathsf{c}^*}_{\mathrm{o},t}^2\right)&+\frac{1}{\sigma}\left(\hat{\mathsf{c}^*}_{\mathrm{y},t}\mathsf{c}_{\mathrm{y},t}+\hat{\mathsf{c}^*}_{\mathrm{m},t}\mathsf{c}_{\mathrm{m},t}+\hat{\mathsf{c}^*}_{\mathrm{o},t}\mathsf{c}_{\mathrm{o},t}\right)+\mathscr{O}_3,\end{aligned}$$

and noting that:

$$\begin{aligned}\frac{c_{\mathrm{y},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{y},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{m},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{m},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{o},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{o},t}^{-\frac{1}{\sigma}}} &= -\frac{1}{2}\frac{1}{\sigma}\left((\mathsf{c}_{\mathrm{y},t}-\hat{\mathsf{c}^*}_{\mathrm{y},t})^2+(\mathsf{c}_{\mathrm{m},t}-\hat{\mathsf{c}^*}_{\mathrm{m},t})^2+(\mathsf{c}_{\mathrm{o},t}-\hat{\mathsf{c}^*}_{\mathrm{o},t})^2\right)\\ &+\frac{3}{1-\frac{1}{\sigma}}+\mathscr{O}_3.\end{aligned}$$

Next, note that the expression for the cost of misallocation from relative price distortions:

$$\Delta_t = \frac{\varepsilon}{2}\frac{\xi}{1-\xi}\mathsf{p}_t'^2+\mathscr{O}_3,$$

and hence that $\Delta_t = \mathscr{O}_2$ since there are no first-order terms. The log-linearization of the price-setting equation implies that:

$$\mathbb{E}_{t-1}\mathsf{p}'_t = \mathscr{O}_2,$$

and hence that:

$$\mathsf{P}'_t = \mathbb{E}_{t-1}\mathsf{P}_t + \mathscr{O}_2, \quad \text{and} \;\; \mathsf{p}'_t = \mathsf{P}_t - \mathbb{E}_{t-1}\mathsf{P}_t + \mathscr{O}_2 = \pi_t - \mathbb{E}_{t-1}\pi_t + \mathscr{O}_2.$$

Therefore, relative price distortions can be written in terms of the squared inflation surprise:

$$\Delta_t = \frac{\varepsilon}{2}\frac{\xi}{1-\xi}(\pi_t - \mathbb{E}_{t-1}\pi_t)^2 + \mathscr{O}_3,$$

The production function implies $\mathsf{Y}_t = \mathsf{A}_t - \Delta_t$. Since $\hat{\mathsf{Y}}_t = \mathsf{A}_t$, it follows that:

$$\mathsf{Y}_t - \hat{\mathsf{Y}}_t = \Delta_t = \mathscr{O}_2.$$

Therefore, $\hat{\mathsf{g}}_t = \mathsf{g}_t + \mathscr{O}_2$. Since g_t is the driving variable for $\mathsf{c}^*_{i,t}$, it follows that:

$$\hat{\mathsf{c}^*}_{i,t} = \mathsf{c}^*_{i,t} + \mathscr{O}_2.$$

Substituting this into the earlier expression:

$$\frac{c_{\mathrm{y},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{y},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{m},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{m},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{o},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{o},t}^{-\frac{1}{\sigma}}} = \frac{3}{1-\frac{1}{\sigma}}-\frac{1}{2}\frac{1}{\sigma}\left((\mathsf{c}_{\mathrm{y},t}-\mathsf{c}^*_{\mathrm{y},t})^2+(\mathsf{c}_{\mathrm{m},t}-\mathsf{c}^*_{\mathrm{m},t})^2+(\mathsf{c}_{\mathrm{o},t}-\mathsf{c}^*_{\mathrm{o},t})^2\right)+\mathscr{O}_3.$$

The results of Proposition 3 imply that:

$$\mathsf{c}_{\mathrm{y},t} - \mathsf{c}^*_{\mathrm{y},t} = \gamma\beta\tilde{\mathsf{d}}_t + \mathscr{O}_2, \quad \mathsf{c}_{\mathrm{m},t} - \mathsf{c}^*_{\mathrm{m},t} = \gamma(\theta-\beta)\tilde{\mathsf{d}}_t + \mathscr{O}_2, \quad \text{and} \;\; \mathsf{c}_{\mathrm{o},t} - \mathsf{c}^*_{\mathrm{o},t} = -\gamma\theta\tilde{\mathsf{d}}_t + \mathscr{O}_2.$$

This leads to:

$$\frac{c_{\mathrm{y},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{y},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{m},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{m},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{o},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{o},t}^{-\frac{1}{\sigma}}} = \frac{3}{1-\frac{1}{\sigma}} - \frac{\gamma^2(\beta^2+(\theta-\beta)\theta)}{\sigma}\tilde{\mathsf{d}}_t^2 + \mathscr{O}_3.$$

Note that:

$$\mathit{\Delta}_t^{-(1-\frac{1}{\sigma})} = 1 - \left(1-\frac{1}{\sigma}\right)\Delta_t + \frac{1}{2}\left(1-\frac{1}{\sigma}\right)^2\Delta_t^2 + \mathscr{O}_3,$$

but since $\Delta_t = \mathscr{O}_2$, this can actually be written as:

$$\mathit{\Delta}_t = 1 - \left(1-\frac{1}{\sigma}\right)\Delta_t + \mathscr{O}_3.$$

By combining equations (??) and (??):

$$\frac{1}{3}\mathit{\Delta}_t^{-(1-\frac{1}{\sigma})}\left\{\frac{c_{\mathrm{y},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{y},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{m},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{m},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{o},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{o},t}^{-\frac{1}{\sigma}}}\right\} = \frac{1}{1-\frac{1}{\sigma}} - \Delta_t - \frac{\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma}\tilde{\mathsf{d}}_t^2 + \mathscr{O}_3.$$

Furthermore we can say that:

$$\aleph^*_t \hat{Y}_t^{1-\frac{1}{\sigma}} = 1 + \mathscr{O}_1 = 1 + \mathscr{I}.$$

It follows that:

$$\frac{1}{3}\hat{\aleph}_t^*\hat{Y}_t^{1-\frac{1}{\sigma}}\Delta_t^{-\left(1-\frac{1}{\sigma}\right)}\left\{\frac{c_{\mathrm{y},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{y},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{m},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{m},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{o},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{o},t}^{-\frac{1}{\sigma}}}\right\}=-\Delta_t-\frac{\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma}\tilde{\mathrm{d}}_t^2+\mathscr{I}+\mathscr{O}_3.$$

By substituting the formula for Δ_t derived earlier:

$$\frac{1}{3}\hat{\aleph}_t^*\hat{Y}_t^{1-\frac{1}{\sigma}}\Delta_t^{-\left(1-\frac{1}{\sigma}\right)}\left\{\frac{c_{\mathrm{y},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{y},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{m},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{m},t}^{-\frac{1}{\sigma}}}+\frac{c_{\mathrm{o},t}^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\hat{c^*}_{\mathrm{o},t}^{-\frac{1}{\sigma}}}\right\}$$
$$=-\frac{\varepsilon}{2}\frac{\xi}{1-\xi}(\pi_t-\mathbb{E}_{t-1}\pi_t)^2-\frac{\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma}\tilde{\mathrm{d}}_t^2+\mathscr{I}+\mathscr{O}_3.$$

Finally, we obtain that:

$$\mathscr{W}_{t_0}=-\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\frac{\varepsilon}{2}\frac{\xi}{1-\xi}(\pi_t-\mathbb{E}_{t-1}\pi_t)^2+\frac{\gamma^2(\beta^2+(\theta-\beta)\theta)}{3\sigma}\tilde{\mathrm{d}}_t^2\right]+\mathscr{I}+\mathscr{O}_3.$$

A.6 Proof of Proposition 11

Substitute the formula for lifetime utility [5.1] into the social welfare function [5.21] and change the order of summation to write the welfare function as

$$W_{t_0}=\mathbb{E}_{t_0}\left[\frac{1}{3}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left\{(\log C_{\mathrm{y},t}+\log C_{\mathrm{m},t}+\log C_{\mathrm{o},t})-\frac{1}{\eta}\left(\frac{H_{\mathrm{y},t}^{\eta}}{\alpha_{\mathrm{y}}^{\eta-1}}+\frac{H_{\mathrm{m},t}^{\eta}}{\alpha_{\mathrm{m}}^{\eta-1}}+\frac{H_{\mathrm{o},t}^{\eta}}{\alpha_{\mathrm{o}}^{\eta-1}}\right)\right\}\right]+\mathscr{I},\quad\text{[A.6.1]}$$

where $\mathscr{I}$ denotes terms independent of monetary policy. Writing $H_{i,t}^{\eta}=H_{i,t}H_{i,t}^{\eta-1}$ and noting that equation [5.3] implies $(H_{i,t}/\alpha_i)^{\eta-1}=w_{i,t}/C_{i,t}$, the terms in the disutility of labour at time t from [A.6.1] can be written as follows:

$$\frac{H_{\mathrm{y},t}^{\eta}}{\alpha_{\mathrm{y}}^{\eta-1}}+\frac{H_{\mathrm{m},t}^{\eta}}{\alpha_{\mathrm{m}}^{\eta-1}}+\frac{H_{\mathrm{o},t}^{\eta}}{\alpha_{\mathrm{o}}^{\eta-1}}=\frac{1}{Y_t}\left(\frac{w_{\mathrm{y},t}H_{\mathrm{y},\mathrm{t}}}{c_{\mathrm{y},t}}+\frac{w_{\mathrm{m},t}H_{\mathrm{m},\mathrm{t}}}{c_{\mathrm{m},t}}+\frac{w_{\mathrm{o},t}H_{\mathrm{o},\mathrm{t}}}{c_{\mathrm{o},t}}\right),\quad\text{[A.6.2]}$$

using the definition $c_{i,t}\equiv C_{i,t}/Y_t$. Next, equation [5.11] is substituted into [A.6.2] to obtain:

$$\frac{H_{\mathrm{y},t}^{\eta}}{\alpha_{\mathrm{y}}^{\eta-1}}+\frac{H_{\mathrm{m},t}^{\eta}}{\alpha_{\mathrm{m}}^{\eta-1}}+\frac{H_{\mathrm{o},t}^{\eta}}{\alpha_{\mathrm{o}}^{\eta-1}}=\left(\frac{w_tN_t}{(1-\tau)Y_t}\right)\left(\frac{\alpha_{\mathrm{y}}}{c_{\mathrm{y},t}}+\frac{\alpha_{\mathrm{m}}}{c_{\mathrm{m},t}}+\frac{\alpha_{\mathrm{o}}}{c_{\mathrm{o},t}}\right).\quad\text{[A.6.3]}$$

Using the particular value of the wage-bill subsidy $\tau=\varepsilon^{-1}$ and $w_t=x_tA_t$ from equation [4.6]:

$$\frac{H_{\mathrm{y},t}^{\eta}}{\alpha_{\mathrm{y}}^{\eta-1}}+\frac{H_{\mathrm{m},t}^{\eta}}{\alpha_{\mathrm{m}}^{\eta-1}}+\frac{H_{\mathrm{o},t}^{\eta}}{\alpha_{\mathrm{o}}^{\eta-1}}=\left(\frac{x_tA_tN_t}{(1-\varepsilon^{-1})Y_t}\right)\left(\frac{\alpha_{\mathrm{y}}}{c_{\mathrm{y},t}}+\frac{\alpha_{\mathrm{m}}}{c_{\mathrm{m},t}}+\frac{\alpha_{\mathrm{o}}}{c_{\mathrm{o},t}}\right).\quad\text{[A.6.4]}$$

Next, noting that [5.10] implies $A_tN_t=\Delta_tY_t$, substitute the expression for real marginal cost x_t from [5.13] into [A.6.4] to obtain:

$$\frac{H_{\mathrm{y},t}^{\eta}}{\alpha_{\mathrm{y}}^{\eta-1}}+\frac{H_{\mathrm{m},t}^{\eta}}{\alpha_{\mathrm{m}}^{\eta-1}}+\frac{H_{\mathrm{o},t}^{\eta}}{\alpha_{\mathrm{o}}^{\eta-1}}=\Delta_t^{\eta}\left(\frac{Y_t}{A_t}\right)^{\eta}\left(c_{\mathrm{y},t}^{\frac{\alpha_{\mathrm{y}}}{3}}c_{\mathrm{m},t}^{\frac{\alpha_{\mathrm{m}}}{3}}c_{\mathrm{o},t}^{\frac{\alpha_{\mathrm{o}}}{3}}\right)\left(\frac{\alpha_{\mathrm{y}}}{c_{\mathrm{y},t}}+\frac{\alpha_{\mathrm{m}}}{c_{\mathrm{m},t}}+\frac{\alpha_{\mathrm{o}}}{c_{\mathrm{o},t}}\right).\quad\text{[A.6.5]}$$

Defining the output gap $\tilde{Y}_t\equiv Y_t/A_t$, it follows that:

$$\frac{1}{3}\frac{1}{\eta}\left(\frac{H_{\mathrm{y},t}^{\eta}}{\alpha_{\mathrm{y}}^{\eta-1}}+\frac{H_{\mathrm{m},t}^{\eta}}{\alpha_{\mathrm{m}}^{\eta-1}}+\frac{H_{\mathrm{o},t}^{\eta}}{\alpha_{\mathrm{o}}^{\eta-1}}\right)=\frac{1}{\eta}\Delta_t^{\eta}\tilde{Y}_t^{\eta}\left(c_{\mathrm{y},t}^{\frac{\alpha_{\mathrm{y}}}{3}}c_{\mathrm{m},t}^{\frac{\alpha_{\mathrm{m}}}{3}}c_{\mathrm{o},t}^{\frac{\alpha_{\mathrm{o}}}{3}}\right)\left(\frac{\alpha_{\mathrm{y}}}{3}c_{\mathrm{y},t}^{-1}+\frac{\alpha_{\mathrm{m}}}{3}c_{\mathrm{m},t}^{-1}+\frac{\alpha_{\mathrm{o}}}{3}c_{\mathrm{o},t}^{-1}\right).\quad\text{[A.6.6]}$$

The terms in consumption:

$$\frac{1}{3}\left(\log C_{\mathrm{y},t}+\log C_{\mathrm{m},t}+\log C_{\mathrm{o},t}\right)=\log \tilde{Y}_t+\frac{1}{3}\left(\log c_{\mathrm{y},t}+\log c_{\mathrm{m},t}+\log c_{\mathrm{o},t}\right)+\mathscr{I}. \qquad [\text{A.6.7}]$$

Given the steady-state values $\bar{\tilde{Y}}=1$ and $\bar{c}_i=1$ for all $i\in\{\mathrm{y},\mathrm{m},\mathrm{o}\}$:

$$\frac{1}{3}\left(\log C_{\mathrm{y},t}+\log C_{\mathrm{m},t}+\log C_{\mathrm{o},t}\right)=\tilde{\mathsf{Y}}_t+\frac{1}{3}(\mathsf{c}_{\mathrm{y},t}+\mathsf{c}_{\mathrm{m},t}+\mathsf{c}_{\mathrm{o},t})+\mathscr{I}. \qquad [\text{A.6.8}]$$

Note that:

$$\frac{1}{3}\left(\mathsf{c}_{\mathrm{y},t}+\mathsf{c}_{\mathrm{m},t}+\mathsf{c}_{\mathrm{o},t}\right)=-\frac{1}{2}\frac{1}{3}\left(\mathsf{c}_{\mathrm{y},t}^2+\mathsf{c}_{\mathrm{m},t}^2+\mathsf{c}_{\mathrm{o},t}^2\right)+\mathscr{O}^3.$$

Consider the second-order approximations:

$$\mathit{\Delta}_t^{\eta}=1+\eta\Delta_t+\frac{\eta^2}{2}\Delta_t^2+\mathscr{O}^3,\quad \text{and}\quad \tilde{Y}_t^{\eta}=1+\eta\tilde{\mathsf{Y}}_t+\frac{\eta^2}{2}\tilde{\mathsf{Y}}_t^2+\mathscr{O}^3.$$

And:

$$c_{\mathrm{y},t}^{\frac{\alpha_{\mathrm{y}}}{3}}c_{\mathrm{m},t}^{\frac{\alpha_{\mathrm{m}}}{3}}c_{\mathrm{o},t}^{\frac{\alpha_{\mathrm{o}}}{3}}=1+\left(\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t}+\frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t}+\frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t}\right)+\frac{1}{2}\left(\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t}+\frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t}+\frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t}\right)^2+\mathscr{O}^3.$$

Also:

$$\frac{\alpha_{\mathrm{y}}}{3}c_{\mathrm{y},t}^{-1}+\frac{\alpha_{\mathrm{m}}}{3}c_{\mathrm{m},t}^{-1}+\frac{\alpha_{\mathrm{o}}}{3}c_{\mathrm{o},t}^{-1}=1-\left(\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t}+\frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t}+\frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t}\right)+\frac{1}{2}\left(\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t}^2+\frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t}^2+\frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t}^2\right)+\mathscr{O}^3.$$

From before:

$$\Delta_t=\frac{\varepsilon\kappa}{2}(\pi_t-\mathbb{E}_{t-1}\pi_t)^2+\mathscr{O}^3.$$

It follows that:

$$\begin{aligned}\frac{1}{\eta}\mathit{\Delta}_t^{\eta}\tilde{Y}_t^{\eta}\left(c_{\mathrm{y},t}^{\frac{\alpha_{\mathrm{y}}}{3}}c_{\mathrm{m},t}^{\frac{\alpha_{\mathrm{m}}}{3}}c_{\mathrm{o},t}^{\frac{\alpha_{\mathrm{o}}}{3}}\right)\left(\frac{\alpha_{\mathrm{y}}}{3}c_{\mathrm{y},t}^{-1}+\frac{\alpha_{\mathrm{m}}}{3}c_{\mathrm{m},t}^{-1}+\frac{\alpha_{\mathrm{o}}}{3}c_{\mathrm{o},t}^{-1}\right)&=\Delta_t+\tilde{\mathsf{Y}}_t+\frac{\eta}{2}\tilde{\mathsf{Y}}_t^2+\frac{1}{2}\frac{1}{\eta}\left(\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t}^2+\frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t}^2+\frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t}^2\right)\\&-\frac{1}{2}\frac{1}{\eta}\left(\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t}+\frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t}+\frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t}\right)^2+\mathscr{I}+\mathscr{O}^3.\end{aligned}$$

Note that:

$$\left(\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t}+\frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t}+\frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t}\right)=\xi\mathsf{d}_t+\mathscr{O}^2.$$

and therefore:

$$\begin{aligned}\left(\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t}^2+\frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t}^2+\frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t}^2\right)-\left(\frac{\alpha_{\mathrm{y}}}{3}\mathsf{c}_{\mathrm{y},t}+\frac{\alpha_{\mathrm{m}}}{3}\mathsf{c}_{\mathrm{m},t}+\frac{\alpha_{\mathrm{o}}}{3}\mathsf{c}_{\mathrm{o},t}\right)^2&=\frac{\alpha_{\mathrm{y}}}{3}(\mathsf{c}_{\mathrm{y},t}-\xi\mathsf{d}_t)^2+\frac{\alpha_{\mathrm{m}}}{3}(\mathsf{c}_{\mathrm{m},t}-\xi\mathsf{d}_t)^2\\+\frac{\alpha_{\mathrm{o}}}{3}(\mathsf{c}_{\mathrm{o},t}-\xi\mathsf{d}_t)^2&=\left(\frac{\alpha_{\mathrm{y}}}{3}(\gamma\beta-\xi)^2+\frac{\alpha_{\mathrm{m}}}{3}(\gamma(\theta-\beta)-\xi)^2+\frac{\alpha_{\mathrm{o}}}{3}(-\gamma\theta-\xi)^2\right)\mathsf{d}_t^2.\end{aligned}$$

And also:

$$\frac{1}{3}\left(\mathsf{c}_{\mathrm{y},t}^2+\mathsf{c}_{\mathrm{m},t}^2+\mathsf{c}_{\mathrm{o},t}^2\right)=\frac{1}{3}\left((\gamma\beta)^2+(\gamma(\theta-\beta)^2+(-\gamma\theta)^2\right)\mathsf{d}_t^2.$$

Now define:

$$\chi\equiv\frac{1}{3}\left((\gamma\beta)^2+(\gamma(\theta-\beta)^2+(-\gamma\theta)^2\right)+\frac{1}{\eta}\left(\frac{\alpha_{\mathrm{y}}}{3}(\gamma\beta-\xi)^2+\frac{\alpha_{\mathrm{m}}}{3}(\gamma(\theta-\beta)-\xi)^2+\frac{\alpha_{\mathrm{o}}}{3}(-\gamma\theta-\xi)^2\right).$$

This can be rearranged as follows:

$$\chi=\frac{\gamma^2}{3}\left(1+\frac{1}{\eta}\right)\left(\theta^2+\beta^2+(\theta-\beta)^2\right)+\frac{1}{3}\frac{1}{\eta}\left((-\gamma\beta)(\gamma\beta)^2+(\gamma(1+\beta))(\gamma(\theta-\beta))^2+(-\gamma)(-\gamma\theta)^2\right)-\frac{\xi^2}{\eta}.$$

Further simplification yields:

$$\chi = \frac{\gamma^2}{3}\left(1 + \frac{1}{\eta}\right)\left(\theta^2 + \beta^2 + (\theta - \beta)^2\right) + \frac{\gamma^3}{3}\frac{1}{\eta}\left((1+\beta)(\theta-\beta)^2 - \theta^2 - \beta^3\right) - \frac{\xi^2}{\eta}.$$

Setting up the Lagrangian for minimizing loss function [5.22] subject to the constraints [5.20a]–[5.20c]:

$$\mathcal{L}_{t_0} = \sum_{t=t_0}^{\infty} \beta^{t-t_0}\mathbb{E}_{t_0}\left[\frac{\varepsilon\kappa}{2}(\pi_t - \mathbb{E}_{t-1}\pi_t)^2 + \frac{\eta}{2}\tilde{\mathsf{Y}}_t^2 + \frac{\chi}{2}\mathsf{d}_t^2\right] + \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\digamma_t\left\{\eta\tilde{\mathsf{Y}}_t + \xi\mathsf{d}_t - \kappa(\pi_t - \mathbb{E}_{t-1}\pi_t)\right\}\right]$$
$$+ \sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[\daleth_t\left\{\lambda\mathsf{d}_t - \mathsf{d}_{t+1}\right\} + \beth_t\left\{\mathsf{i}_{t-1} + \theta\mathsf{d}_t + \mathsf{d}_{t-1} - \tilde{\mathsf{Y}}_t + \tilde{\mathsf{Y}}_{t-1} - \pi_t - \hat{\mathsf{r}}_t^*\right\}\right]. \quad \text{[A.6.9]}$$

The first-order conditions of [A.6.9] with respect to the endogenous variables π_t, d_t, $\tilde{\mathsf{Y}}_t$, and i_t are:

$$\varepsilon\kappa(\pi_t - \mathbb{E}_{t-1}\pi_t) - \kappa(\digamma_t - \mathbb{E}_{t-1}\digamma_t) - \beth_t = 0; \quad \text{[A.6.10a]}$$
$$\chi\mathsf{d}_t + \xi\digamma_t + \lambda\daleth_t - \beta^{-1}\daleth_{t-1} + \theta\beth_t + \beta\mathbb{E}_t\beth_{t+1} = 0; \quad \text{[A.6.10b]}$$
$$\eta\tilde{\mathsf{Y}}_t + \eta\digamma_t - \beth_t + \beta\mathbb{E}_t\beth_{t+1} = 0; \quad \text{[A.6.10c]}$$
$$\beta\mathbb{E}_t\beth_{t+1} = 0. \quad \text{[A.6.10d]}$$

Using [A.6.10d], it follows from [A.6.10c] that:

$$\digamma_t = -\tilde{\mathsf{Y}}_t + \frac{1}{\eta}\beth_t. \quad \text{[A.6.11]}$$

The Phillips curve [5.20c] can be rearranged as follows:

$$\tilde{\mathsf{Y}}_t = \frac{\kappa}{\eta}(\pi_t - \mathbb{E}_{t-1}\pi_t) - \frac{\xi}{\eta}\mathsf{d}_t, \quad \text{[A.6.12]}$$

and combining this with [A.6.11] yields:

$$\digamma_t = \frac{\xi}{\eta}\mathsf{d}_t - \frac{\kappa}{\eta}(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{1}{\eta}\beth_t. \quad \text{[A.6.13]}$$

Taking conditional expectations and using [A.6.10d]:

$$\mathbb{E}_t\digamma_{t+1} = \frac{\xi}{\eta}\mathbb{E}_t\mathsf{d}_{t+1}. \quad \text{[A.6.14]}$$

Now multiply [A.6.10b] by β and use [A.6.10d] to deduce:

$$\daleth_t = \beta\lambda\mathbb{E}_t\daleth_{t+1} + \beta\mathbb{E}_t\left[\chi\mathsf{d}_{t+1} + \xi\digamma_{t+1}\right]. \quad \text{[A.6.15]}$$

Substituting from [A.6.14] implies:

$$\daleth_t = \beta\lambda\mathbb{E}_t\daleth_{t+1} + \beta\left(\chi + \frac{\xi^2}{\eta}\right)\mathbb{E}_t\mathsf{d}_{t+1}. \quad \text{[A.6.16]}$$

Solving forwards and using [5.20a] to deduce that $\mathbb{E}_t\mathsf{d}_{t+\ell} = \lambda^\ell\mathsf{d}_t$ yields:

$$\daleth_t = \left(\chi + \frac{\xi^2}{\eta}\right)\left(\frac{\beta\lambda}{1-\beta\lambda^2}\right)\mathsf{d}_t. \quad \text{[A.6.17]}$$

Next, substitute the expressions for $\digamma_t$ and $\daleth_t$ from [A.6.13] and [A.6.17] into [A.6.10b] and use [A.6.10d]:

$$\chi \mathsf{d}_t + \xi\left(\frac{\xi}{\eta}\mathsf{d}_t - \frac{\kappa}{\eta}(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{1}{\eta}\beth_t\right) + \left(\chi + \frac{\xi^2}{\eta}\right)\left(\frac{\beta\lambda}{1-\beta\lambda^2}\right)\left(\lambda \mathsf{d}_t - \frac{1}{\beta}\mathsf{d}_{t-1}\right) + \theta\beth_t = 0. \quad \text{[A.6.18]}$$

Simplifying this equation yields:

$$\left(\chi + \frac{\xi^2}{\eta}\right)\left(\frac{1}{1-\beta\lambda^2}\right)(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t) - \frac{\xi\kappa}{\eta}(\pi_t - \mathbb{E}_{t-1}\pi_t) + \left(\theta + \frac{\xi}{\eta}\right)\beth_t = 0, \quad \text{[A.6.19]}$$

where equation [5.20a] has been used to write $\lambda \mathsf{d}_{t-1} = \mathbb{E}_{t-1}\mathsf{d}_t$.

Use equations [A.6.13] and [A.6.14] to deduce:

$$\digamma_t - \mathbb{E}_{t-1}\digamma_t = \frac{\xi}{\eta}(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t) - \frac{\kappa}{\eta}(\pi_t - \mathbb{E}_{t-1}\pi_t) + \frac{1}{\eta}\beth_t. \quad \text{[A.6.20]}$$

This equation can be combined with [A.6.10a] to yield:

$$\left(1 + \frac{\kappa}{\eta}\right)\beth_t = \kappa\left(\left(\varepsilon + \frac{\kappa}{\eta}\right)(\pi_t - \mathbb{E}_{t-1}\pi_t) - \frac{\xi}{\eta}(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t)\right). \quad \text{[A.6.21]}$$

Multiplying both sides of [A.6.19] by $(1 + \kappa/\eta)$:

$$\left(\chi + \frac{\xi^2}{\eta}\right)\left(1 + \frac{\kappa}{\eta}\right)\left(\frac{1}{1-\beta\lambda^2}\right)(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t) - \frac{\xi\kappa}{\eta}\left(1 + \frac{\kappa}{\eta}\right)(\pi_t - \mathbb{E}_{t-1}\pi_t) + \left(\theta + \frac{\xi}{\eta}\right)\left(1 + \frac{\kappa}{\eta}\right)\beth_t = 0. \quad \text{[A.6.22]}$$

Now substituting [A.6.21] into this equation:

$$\begin{aligned}&\left(\left(\chi + \frac{\xi^2}{\eta}\right)\left(1 + \frac{\kappa}{\eta}\right)\left(\frac{1}{1-\beta\lambda^2}\right) - \frac{\xi\kappa}{\eta}\left(\theta + \frac{\xi}{\eta}\right)\right)(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t) \\ &+\kappa\left(\left(\theta + \frac{\xi}{\eta}\right)\left(\varepsilon + \frac{\kappa}{\eta}\right) - \frac{\xi}{\eta}\left(1 + \frac{\kappa}{\eta}\right)\right)(\pi_t - \mathbb{E}_{t-1}\pi_t) = 0.\end{aligned} \quad \text{[A.6.23]}$$

Note that:

$$\kappa\left(\left(\theta + \frac{\xi}{\eta}\right)\left(\varepsilon + \frac{\kappa}{\eta}\right) - \frac{\xi}{\eta}\left(1 + \frac{\kappa}{\eta}\right)\right) = \kappa\left(\theta\left(\varepsilon + \frac{\kappa}{\eta}\right) + (\varepsilon - 1)\frac{\xi}{\eta}\right).$$

And:

$$\left(\left(\chi + \frac{\xi^2}{\eta}\right)\left(1 + \frac{\kappa}{\eta}\right)\left(\frac{1}{1-\beta\lambda^2}\right) - \frac{\xi\kappa}{\eta}\left(\theta + \frac{\xi}{\eta}\right)\right) = \left(\chi + \frac{\xi^2}{\eta}\right)\left(1 + \left(1 + \frac{\kappa}{\eta}\right)\left(\frac{\beta\lambda^2}{1-\beta\lambda^2}\right)\right) + \frac{\kappa}{\eta}(\chi - \xi\theta).$$

It follows that:

$$\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t = -\left\{\frac{\kappa\left(\theta\left(\varepsilon + \frac{\kappa}{\eta}\right) + (\varepsilon - 1)\frac{\xi}{\eta}\right)}{\left(\chi + \frac{\xi^2}{\eta}\right)\left(1 + \left(1 + \frac{\kappa}{\eta}\right)\left(\frac{\beta\lambda^2}{1-\beta\lambda^2}\right)\right) + \frac{\kappa}{\eta}(\chi - \xi\theta)}\right\}(\pi_t - \mathbb{E}_{t-1}\pi_t).$$

Calculating the surprise component:

$$(\pi_t - \mathbb{E}_{t-1}\pi_t) + (\mathsf{g}_t - \mathbb{E}_{t-1}\mathsf{g}_t) = \theta(\mathsf{d}_t - \mathbb{E}_{t-1}\mathsf{d}_t).$$

The over-weighting of the price level in the weighted nominal income target is:

$$\varpi = \frac{\kappa\theta\left(\theta\left(\varepsilon + \frac{\kappa}{\eta}\right) + (\varepsilon - 1)\frac{\xi}{\eta}\right)}{\left(\chi + \frac{\xi^2}{\eta}\right)\left(1 + \left(1 + \frac{\kappa}{\eta}\right)\left(\frac{\beta\lambda^2}{1-\beta\lambda^2}\right)\right) + \frac{\kappa}{\eta}(\chi - \xi\theta)}.$$